Geometries of Strongly Minimal Sets

I: Geometries and pregeometries

Def^n: A pregeometry is a set X with a map $cl: P(X) \to P(X)$ s.t.

- $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
- $cl(cl(A)) = cl(A)$
- $A \subseteq cl(A)$

} cl is a "closure operator"

A closed if $A = cl(A)$

- $b \in cl(Ac) \setminus cl(A) \Rightarrow c \in cl(Ab)$ "Exchange" ($cl(Ac) := cl(A \cup \{c\})$)
- $cl(A) = \bigcup_{A_0 \subseteq_{fin} A} cl(A_0)$ "Finite character"

(finite pregeoms are also called "matroids")

Example:

- X = vector space $/K$, $cl(A) := \langle A \rangle_K$ [s/X/V/g]
  Exchange is Steinitz' Theorem
- Note $\langle A \rangle_K = acl(A)$ in the s.m. structure $(X; +, 0, (k \cdot)_{k \in K})$ (assuming X infte)

~~Lemma: Let M be s.m. Then (M; acl^M) is a pregeom.~~
~~Pf: All immediate except:~~
~~· acl(acl(A)) = acl(A): exercise~~
~~· Exchange~~

Recall: An infte structure M is strongly minimal if every def^{ble} subset of any elem^y ext^n is finite or cofinite.

Equivalently: for any $A \subseteq M' \succeq M$, there is a unique non-alg^c type $p_A \in S(A)$

Lemma: If M is s.m., then $(M; acl^M)$ is a pregeom.

Pf: All immediate except:

- $acl(acl(A)) = acl(A)$: Exercise
- Exchange: suppose $b \in acl(Ac) \setminus acl(A)$ but $c \notin acl(Ab)$

  Say $\vDash \phi(b,c)$ and $\vDash \exists^{\leq n} x . \phi(x,c)$ ϕ / A

  WLOG $\vDash \forall y . \exists^{\leq n} x . \phi(x,y)$ (1)

  Since $c \notin acl(Ab)$, $\phi(b, M)$ is cofinite

  WLOG $\vDash \forall x . (\exists y . \phi(x,y) \to \exists^{\leq \ell} y . \neg\phi(x,y))$ (2)

  Now $\exists y . \phi(M, y)$ is cofinite since $b \notin acl(A)$

  so let $b_1, \ldots, b_{n+1}$ distinct s.t. $\vDash \exists y . \phi(b_i, y)$

  By (2), exists c' s.t. $\vDash \phi(b_i, c')$ $\forall i$

  contradicting (1). $\square$

Def^n: Let (X, cl) be a pregeom, $A, B \subseteq X$.

- A basis for A over B is a min^l subset $A' \subseteq A$ s.t. $cl(A'B) = cl(AB)$.
- $\dim(A/B) := |A'|$ for any basis A' for A over B

  Fact: well-defined
- $\dim(A), \dim(B) < \infty \Rightarrow \dim(A/B) = \dim(AB) - \dim(B)$
- $A \underset{C}{\perp} B$ means: ~~$\dim(A/BC) = \dim(A/C)$~~ Any basis for A over C is also a basis for A over BC.

  equiv (if $\dim(A/C) < \infty$): $\dim(A/C) = \dim(A/BC)$
- By exchange, $A \underset{C}{\perp} B \Leftrightarrow B \underset{C}{\perp} A$

Def^n: (X, cl) is homogeneous if for any closed $C \subseteq X$ and $a, b \in X \setminus C$ exists automorphism σ of (X, cl) s.t.

$\sigma(c) = c$ $\forall c \in C$

$\sigma(a) = b$

For M s.m., (M, acl) is homog^s, witnessed by $\sigma \in Aut(M)$.

Def^n: A geometry is a pregeom^y s.t. $cl(\emptyset) = \emptyset$, $cl(a) = \{a\}$ $\forall a \in X$

- The associated geometry of a pregeom^y (X, cl) is $(\{cl(a) : a \in X\}; cl')$ where $cl(a) \in cl'(B)$ iff $a \in cl(\bigcup B)$.

A projective geom^y is the associated geom^y of the pregeom^y defined above of a vector space over a division ring.

II: Trichotomy

Def^n: (X, cl) is degenerate (or trivial) if $cl(A) = \bigcup_{a \in A} cl(a)$ $\forall A \subseteq X$ (equiv: in the geom, $cl = id$)

Example: acl in pure set

Def^n: (X, cl) is modular if for any fin. dim closed $A, B \subseteq X$,

$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B)$

equiv: $\dim(A/B) = \dim(A/A \cap B)$ i.e. $A \underset{A \cap B}{\perp} B$

Proj^{ve} geom^{ies} are modular. Conversely,

Fact [Veblen–Young]: Any non-degenerate modular homog^s geom^y of dim^n ≥ 4 is isomorphic to a proj^{ve} geom^y.

Example: V VS/K admits a second natural pregeom^y: take the closed sets as the cosets of the subspaces. This is the affine geom^y of V.

This is not modular: consider two parallel lines in a common plane.

But if we localise at a point, we obtain the proj^{ve} pregeom^y

Def^n: The localisation of (X, cl) at $A \subseteq X$ is (X, cl_A) where $cl_A(B) := cl(AB)$.

This corresponds to adding constants to a structure: if $(X, cl) = (M, acl^M)$, then $cl_A = acl^{M_A}$.

Def^n: A pregeometry (X, cl) is locally modular if the localisation at any $a \in X \setminus cl(\emptyset)$ is modular

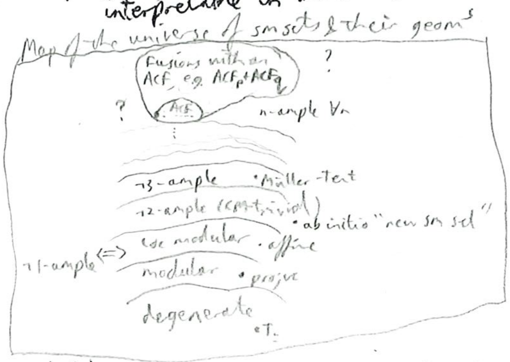
(we will see that in the s.m. case, this is equiv to: some localisation is modular)

Prop^n: Let $K \models ACF$.
Then $(K; acl)$ is not loc. mod.

pf: Let $c \in K \setminus acl(\emptyset)$, and let a, b, x ind^{nt} / c
i.e. $\dim(c,a,b,x) = 4$.
Set $y := ax + b$.
Sps loc. mod. Then since $ab \not\perp_c xy$,
exists $d \in (acl(cab) \cap acl(cxy)) \setminus acl(c)$

$x \notin acl(c,d)$ since $x \perp abc$
so $y \in acl(c,d,x)$.
So say $p(x,y) = 0$, $p \in acl(c,d)[X,Y]$ irr^{ble}.
Then $(Y - aX - b) \mid p$, and p is irr^{ble} in $acl(cab)[X,Y]$ ($\supseteq acl(cd)$)
so $p = \alpha(Y - aX - b)$ some $\alpha \in acl(cab)$.
But then $\alpha \in acl(cd)$, so $a, b \in acl(cd)$ $\times$ $\dim(ab/c) = 2$ $\square$

Zilber Trichotomy Conj^{re}
The geometry of any s.m. set is either
(i) degenerate
(ii) affine or proj^{ve} geom^y } loc. mod.
(iii) the geometry of an ACF
Moreover in case (iii) an ACF is interpretable in the s.m. structure.



(part of)
Zilber's Weak Trichotomy Th^m:
Any s.m. str^{re} is loc. mod.
iff it interprets no pseudoplane

Def^n: A pseudoplane is a binary rel^n
$I \subseteq P \times L$
s.t. pI, $I\ell$ inf^{te} $\forall p, \ell$
$pI \cap p'I$, $I\ell \cap I\ell'$ finite
$\forall p \neq p'$, $\ell \neq \ell'$

e.g. points and lines in proj^{ve} plane $\mathbb{P}^2(K)$
("completion" of $y = ax + b$ example)

III Imaginaries

- For M a str^{re},
  M^{eq} is the exp^n by
  a new sort $M^n/\sim$
  and the quotient f^n $f_\sim : M^n \to M^n/\sim$
  for each $\emptyset$-def^{ble} eq^{nce} rel^n $\sim \subseteq M^n \times M^n$.
- "interpretable in M" means: definable in M^{eq}.
- $T^{eq} := Th(M^{eq})$ where $T = Th(M)$
- acl^{eq} := acl in T^{eq}
- dcl^{eq} " dcl " "
- T has (weak/geometric) elimination of imag^s
  if for any $M \models T$ and $a \in M^{eq}$ exists $b \in M^{<\omega}$ s.t.
  EI: $dcl^{eq}(a) = dcl^{eq}(b)$
  gEI: $acl^{eq}(a) = acl^{eq}(b)$
  wEI: $dcl^{eq}(a) \subseteq dcl^{eq}(b) \subseteq acl^{eq}(a)$
  ($EI \Rightarrow wEI \Rightarrow gEI$)

Prop^n: Any s.m. T with $acl(\emptyset)$ inf^{te} has wEI

pf: Let $M \models T$ and $a \in M^{eq}$.
$a = c/\sim$ say, $c \in M^n$
$f_\sim^{-1}(a) = \{x ; x/\sim = a\} \subseteq M^n$ is def^{ble} / a, so STS
claim: If $\emptyset \neq X \subseteq M^n$ is def^{ble} / $a \in M^{eq}$
then $X \cap acl^{eq}(a) \neq \emptyset$
pf: Let $\pi : M^n \to M$ be a co-ord^{te} proj^n.
If $\pi(X)$ is finite, let $b_0 \in \pi(X)$.
Else, $\pi(X)$ is cofinite, so let $b_0 \in \pi(X) \cap acl(\emptyset)$.
Then $b_0 \in acl^{eq}(a)$. If $n = 1$, we conclude.
$n > 1$: $\pi^{-1}(b_0) \cap X$ is def^{ble} / ab_0
so by IH $\exists h \in (\pi^{-1}(b_0) \cap X) \cap acl^{eq}(ab_0)$ ($\subseteq acl^{eq}(a)$) $\square$ $\square$

end of seminar

So for $a \in M^{eq}$, can define $\dim(a) := \dim(\tilde{a}/C)$
where, if $a = b/\sim$, $C \subseteq M$ is inf^{te} and $C \perp b$,
and $acl^{eq}(\tilde{a}C) = acl^{eq}(aC)$.

Def^n: T arb^y, $\mathbb{M} \models T$ monster,
$c \subseteq \mathbb{M}^{eq}$ is a canonical base for $p \in S(\mathbb{M})$
(we write "$cb(p) = c$")
if $\forall \sigma \in Aut(\mathbb{M}).\ (\sigma p = p \Leftrightarrow \sigma|_c = id|_c)$
equiv. $dcl^{eq}(c) = Fix(Aut(\mathbb{M}^{eq}/p))$

Fact:
- If M is ω-stable, $\exists \alpha \in \mathbb{M}^{eq}$. $cb(p) = \alpha$
1. For $C \subseteq B$, $a \perp_C B$ iff $cb(a/B) \subseteq acl^{eq}(C)$
   where $cb(a/B) := cb(tp(a'/\mathbb{M}))$ where $a' \equiv_{acl^{eq}(B)} a$ and $a' \perp_B \mathbb{M}$