A new str. min. set, Part 2

The goal of this talk is to present Hrushovskis counterexample for Zilbers conjecture, that a strongly minimal theory must be either locally modular or interpret an infinte field. I follow closely Zieglers "An exposition of Hrushovskis new strongly minimal set" and in the last section Hrushovskis "A new strongly minimal set".
The talk is based on Zhengqing Hes talk, so first I will repeat the setting, some important definition and statements and some basic lemmata, which follow from these. Later on I will just refer to these, so it does not prolong some proofs unneccessarily. Then we will proceed to the construction of the counterexample and show its important properties.
Just some remarks before we start, for better legibility we will sometimes use AB instead of $A \cup B$.

0.1 Repition and basics

One corollary we need from Blaise Boissoneaus talk

Corollary 0.1. $a \mathop{\smile\!\!\!\!\!\!|}_C B$ iff $\mathrm{Cb}(p) \subseteq \mathrm{acl}^{eq}(C)$.

Now we repeat some parts of Zhengqing Hes talk. These statements we do not proof and refer to her notes. We will also take a look at some easy lemma, which we will prove here instead of extending later proofs unnecessarily:

The setting: We consider a language L with just one ternary relation symbol R and $\mathcal{C}$ is the class of all L-structures $M = (M, R^M)$, where R^M is irreflexive and symmetric.

Definition 0.2. We defined δ on a given structure M as $\delta(A) = |A| - |R(A)|$ and $\delta(A/B) = \delta(A \cup B) - \delta(B)$ for sets A, B.
A finite subset A is closed in M, or M is a strong extension of A , $A \leq M$, if $\delta(A) \leq \delta(B)$ for all $A \subseteq B \subseteq M$.
We define $\mathcal{C}^0 = \{M \in \mathcal{C} | \emptyset \leq M\}$.

Example 0.3. Define $C_{nm} = \{a_1, a_2, b_1, b_2, c\}$ with $R(C_{nm})$ consisting of $\{a_1, b_1, c\}$ and $\{a_2, b_2, c\}$. C_{nm} belongs to $\mathcal{C}^0$.

Definition 0.4. The closure $\mathrm{cl}(X)$ of a set X in M is the smallest closed set containing X.

Lemma 0.5. The closure of a finite set is finite.

Remark 0.6. $\delta(A/B) = \delta(A \cup B) - \delta(B) = |A \cup B| - |R(A \cup B)| - (|B| - |R(B)|) = |A \setminus B| - |R(A \cup B) \setminus R(B)|$.
In particular, for A and B disjoint we get: $\delta(A/B) = |A| - |R(A \cup B) \setminus R(B)|$

Lemma 0.7. For sets A disjoint to $B \subseteq M$, if all elements of M, which are connected to A lie in B, then $\delta(A/M) = \delta(A/B)$.

Lemma 0.8. If $A_0, ..., A_n$ are disjoint subsets, then for any set M we have:

$$\delta(A_0...A_n/M) = \sum_{k=0}^{n} \delta(A_k...A_n/(A_{k+1}...A_n M).$$

Lemma 0.9. If $|R(B \cup M)| \geq |R(B)| + |R(M)| - |R(B \cap M)| + r$, then $\delta(B/M) \leq \delta(B/B \cap M) - r$.

Proof. If $|R(B \cup M)| \geq |R(B)| + |R(M)| - |R(B \cap M)| + r$, then equivalently $|R(B \cup M) \setminus R(M)| \geq r - |R(B \cap M) \setminus R(B)|$ and hence

$$\delta(B/M) = |B \setminus M| - |R(B \cup M) \setminus R(M)| \leq |B \setminus (M \cap B)| + |R(B \cap M) \setminus R(B)| - r = \delta(B/B \cap M) - r$$

□

Lemma 0.10. Let X be a subset of Y. Then

$$X \leq Y \Leftrightarrow \delta(A/A \cap X) \geq 0 \text{ for all finite} A \subseteq Y.$$

Lemma 0.11. If $A \leq C, A \subseteq B \subseteq C$, then $A \leq B$.

Lemma 0.12. Let A, B be any sets in M. Then

$$A \cup B \leq M \text{ iff } A' \cup B' \leq M \text{ for every finite } A' \leq A, B' \leq B.$$

Proof. "$\Leftarrow$"Follows directly from lemma 0.11, since $A' \cup B' \subseteq A \cup B \subseteq M$.
"$\Rightarrow$"Assume there exists some finite $A' \leq A, B' \leq B$ with $A' \cup B' \nleq M$- Then by lemma 0.10 there exists a finite $D \subseteq M$ such that $\delta(D/D \cap (A'B')) < 0$. By submodularity we get $\delta(D/D \cap (AB)) \leq \delta(D/D \cap (A'B')) < 0$. And again by lemma 0.10 this implies that $A \cup B \nleq M$. □

Definition 0.13. An extension $B \leq C$ is minimal if B is a maximal proper closed subset of C.

Lemma 0.14. A proper strong extension C of B is minimal iff $\delta(C/D) < 0$ for all $B \subsetneq D \subsetneq C$.
We will also use a slightly different version of this lemma:
A proper strong extension $B \leq B \sqcup A$ is minimal iff $\delta(B \cup A/B \cup D) < 0$ for all $D \subsetneq A$.

Corollary 0.15. If $B \leq C$ is minimal and C is neither contained in X nor disjoint from X, then we have $\delta(C/X \cup B) < 0$.

Lemma 0.16. If $B \leq C$ is minimal, there are two cases

1. $\delta(C/B) = 1$ and $C = \{B \cup \{c\}$.
2. $\delta(C/B) = 0$.

Definition 0.17. The dimension of A is defined as

$$d(A) = \min\{\delta(B) | A \subseteq B\} = \delta(\text{cl}(A)).$$

The corresponding Cl (for which d is the dimension function) we call the geometric closure.

Remark 0.18. Note that $d(A) \leq \delta(A)$ and $\text{cl}(X) \subseteq \text{Cl}(X)$. Furthermore, $d(A) = d(\text{cl}(A))$.

Remark 0.19. If C is a subset of M, B closed in C and $\delta(C/B) = 0$, then C is contained in $\text{Cl}(B)$.

Lemma 0.20. If X is closed in M, then $\text{Cl}(X)$ is the union of all extensions C with $\delta(C/X) = 0$.

Proof. Since $X \leq M$ for any C with $X \subseteq C \subseteq M$ we have B is closed in C. Now if C fulfilled $\delta(C/X) = 0$, then by the remark $C \subseteq \text{Cl}(B)$.
On the other hand, if we have any $x \in \text{Cl}(X)$, then $0 = d(x/X) = \delta(\text{cl}(X \cup \{x\})/\text{cl}(X)) = \delta(\text{cl}(X \cup \{x\})/X)$. Hence $x \in \text{cl}(X \cup \{x\})$ lies in the union. □

Lemma 0.21. If $d(c/B) = 1$, then c is not connected to $\text{cl}(B)$ and $\text{cl}(B) \cup \{c\}$ is closed.

Proof. If $c \notin \text{cl}(B)$, but c is connected to $\text{cl}(B)$, then

$$\delta(\text{cl}(B) \cup \{c\}) = |\text{cl}(B)|| + 1 - |R(\text{cl}(B) \cup \{c\})| \leq |\text{cl}(B)|| + 1 - (|R(\text{cl}(B))| + 1) = \delta(\text{cl}(B)).$$

and hence we get

$$d(c/B) = d(B \cup \{c\})) - d(B) = \min\{\delta(A) | B \cup \{c\} \subseteq A\} - \delta(\text{cl}(B)) \leq \delta(\text{cl}(B) \cup \{c\}) - \delta(\text{cl}(B)) \leq 0.$$

And if c is not connected to $\mathrm{cl}(B)$ and $\mathrm{cl}(B) \cup \{c\}$ is not closed, then exists some $\mathrm{cl}(B) \cup \{c\} \subseteq A$ with $\delta(A) < \delta(\mathrm{cl}(B) \cup \{c\})$ and hence

$$d(c/B) = \min\{\delta(A')|Bc \subseteq A'\} - d(B) \leq \delta(A) - d(B) < \delta(\mathrm{cl}(B) \cup \{c\}) - \delta(\mathrm{cl}(B)) = 1.$$

□

Definition 0.22. Let A, B be sets in M, then $A \cup B$ is the free amalgam $A \otimes_C B$ iff $A \cap B = C$ and $R(A \cup B) = R(A) \cup R(B)$.

Lemma 0.23. For sets A, B in M we have:

$$A \cup B = A \otimes_C B \text{ iff. } A' \cup B' = A' \otimes_C B' \text{ for all finite } A' \leq A, B' \leq B \text{ with } A' \cap B' = C.$$

Proof. "⇒" This is clear by definition.
"⇐"Assume $A \cup B$ is not the free amalgam. Hence there exist $a \in A, b \in B$ such that $R(a, b, c)$ for some $c \in A \cup B$ holds. W.l.o.g. $c \in A$. Now consider $A' = \mathrm{cl}(a, c) \leq A$ and $B' = \mathrm{cl}(b)$. By lemma 0.5 A', B' are finite and they are not the free amalgam. □

Chapter 1

The collapse

Definition 1.1. A pair A/X of disjoint sets is called *prealgebraic minimal* if

(i) $X \cup A$ belongs to $\mathcal{C}^0$.

(ii) $X \leq X \cup A$ is minimal.

(iii) $\delta(A/X) = 0$.

A prealgebraic minimal pair A/B is called *good* if $\delta(A/B') > 0$ for every proper subset $B' \subsetneq B$.

Proposition 1.2. For every prealgebraic pair A/X there exists a unique $B \subseteq X$ such that A/B is good. We call this B the basis of A/X.

Proof. Let B be the set of all $x \in X$ which are connected with an element of A i.e. there exists an $a \in A$ and $y \in X \cup A$ such that $R(x, a, y)$. By definition we then get $R(X \cup A') = R(X) \sqcup R(B \cup A') \setminus R(B)$ for any $A' \subseteq A$ and hence

$$|R(X \cup A')| = |R(X)| + |R(B \cup A') \setminus R(B)| = |R(X)| + |R(B \cup A')| - |R(B)|.$$

A/B is pralgebraic minimal, since:

(i) $B \cup A$ belongs to $\mathcal{C}^0$, because $X \cup A$ does.

(ii) $B \leq B \cup A$ is minimal: Using corrollary 0.14 to show minimality, take any $A' \subsetneq A$:

$$\begin{aligned}\delta(B \cup A/B \cup A') &= |A \setminus A'| - |R(B \cup A) \setminus R(B \cup A')| \\ &= |A \setminus A'| - (|R(B \cup A)| - |R(B)| + |R(X)|) + (|R(B' \cup A')| - |R(B)| + |R(X)|) \\ &= |A \setminus A'| - |R(X \cup A)| + |R(X \cup A')| = \delta(X \cup A/X \cup A') \overset{A/X \text{ min}}{<} 0\end{aligned}$$

(iii)

$$\delta(A/B) = |A| - |R(A \cup B) \setminus R(B)| = |A| - |R(X \cup A) \setminus R(X)| = \delta(A \setminus X) = 0$$

Now goodness follows since for any subset $B' \subsetneq B$, there is at least one element $b \in B \setminus B'$. Now b is connected to A and hence $|R(A \cup B') \setminus R(B')| = |R(A \cup B') \setminus R(B)| < |R(A \cup B) \setminus R(B)|$. Therefore

$$\delta(A/B') = |A| - |R(A \cup B') \setminus R(B')| > |A| - |R(A \cup B)| = \delta(A/B) = 0$$

Now assume we have another good pair A/C with $x \notin B$, then x is not connected to A and we have $|R(A \cup (C \setminus x)) \setminus R(C \setminus x)| = |R(A \cup C) \setminus R(C)|$. Hence $\delta(A/C \setminus x) = |A| - |R(A \cup (C \setminus x)) \setminus R(C \setminus x)| = |A| - |R(A \cup C) \setminus R(C)| = \delta(A/C) = 0$, which contradicts the goodness of A/C. Therefore C must lie in B, but by goodness it cannot be a proper subset. Hence we have equality.

□

Lemma 1.3. For a prealgebraic minimal pair A/X with basis B we get:

a) $X \cup A = X \otimes_B (B \cup A)$.

b) $|B| \leq 2 \cdot |A|$.

Proof. a) As sets we clearly get $X \otimes_B (B \cup A) = X \cup (B \cup A) = X \cup A$. Now since all $x \in X$ which are connected with A lie in B we also have $R_\otimes(X \cup A) = R(X) \cup R(B \cup A) = R(X \otimes_B (B \cup A))$.

b) $0 = \delta(A/B) = |A| - (|R(A \cup B)| - |R(B)|)$ implies $R' = R(B \cup A) \setminus R(B)$ has $|A|$ elements. By goodness (or the characterization of the basis as above) every element of B belongs to some set in R', but such a set contains at most 2 elements of B. Hence $|A| = |R'| \geq \frac{1}{2}|B|$.

□

Definition 1.4. A *code* α is the isomorhism type of a good pair A_α/B_α.
A *pseudo Morley sequence* of α over B is a pairwise disjoint sequence $A_0, A_1, ...$ such that all A_i/B are of type α.

Lemma 1.5. Let $M \leq N$ be in $\mathcal{C}^0$. If N contains a pseudo Morley sequence (A_i) of α over B with more than $\delta(B)$ elements, then one of the following occurs:

1. $B \subseteq M$
2. Some A_i lies in $N \setminus M$.

Proof. Assume $A_0, ..., A_{r-1}$ lie in M and $A_r, ..., A_{r+s-1}$ are neither in M nor in $N \setminus M$. Further assume B is not contained in M. Since A_i/B is good, each A_i contains at least one element which is connected to B. Hence

$$|R(B \cup M)| \geq |R(B)| + |R(M)| - |R(B \cap M)| + r.$$

So we get

$$\delta(B/M) \overset{0.9}{\leq} \delta(B/B \cap M) - r = \delta(B) - \delta(B \cap M) - r \overset{M \in \mathcal{C}^0}{\leq} \delta(B) - r.$$

By the minimality of A_i/B, corollary 0.15 implies $\delta(A_i/A_r...A_{i-1}MB) < 0$ or equiv. $\delta(A_i/A_r...A_{i-1}MB) \leq -1$ for all $i \in [r, r+s-1]$. Therefore

$$\delta(A_r....A_{r+s-1}/MB) \overset{(0.8)}{=} \delta(A_r...A_{r+s-1}/A_r...A_{r+s-2}MB) + \delta(A_r...A_{r+s-2}/A_r...A_{r+s-3}MB) + ... + \delta(A_r/MB) \leq -s.$$

Or equiv. $\delta(A_r....A_{r+s-1}MB) \leq \delta(MB) - s$. This implies

$$0 \overset{M \leq N}{\leq} \delta(A_r...A_{r+s-1}B/M) = \delta(A_r...A_{r+s-1}MB) - \delta(M) \leq \delta(MB) - s - \delta(M) = \delta(B/M) - s \leq \delta(B) - r - s.$$

This contradicts the pre-condition $r + s \geq \delta(B)$ □

For every code α we now fix a natural number $\mu(\alpha) \geq \delta(B_\alpha)$.

Definition 1.6. A pseudo Morley sequence of length $> \mu(\alpha)$ is called a *long* pseudo Morley sequence.
Let $\mathcal{C}^\mu$ be the class of all $M \in \mathcal{C}^0$ without any long pseudo Morley sequences.

Example 1.7.

- $C_{nm} \in \mathcal{C}^\mu$.

  Up to isomorphism the only two good pairs are c/a_1b_1 and b_2/a_2c. The only pseudo Morley sequences over their isomorphism types are of length one, which implies not long.

- $F_n \in \mathcal{C}^\mu$ for any $n < \omega$.

  If we had any good pair A/B, then $0 = \delta(A/B) = |A| - |R(A \cup B) \setminus R(B)| \overset{\text{no rel.}}{=} |A|$. So there is no good pairs and no (long) pseudo Morley sequences.

- If $M \in \mathcal{C}^\mu$ and we add a new unconnected point c to M, then $M' := M \cup \{c\} \in \mathcal{C}^\mu$.

  For any good pair A/B in M', by the minimality c cannot be in A and by goodness c cannot be in B. Hence any pseudo Morley sequence in M' is also a pseudo Morley sequence in M and hence not long.

Lemma 1.8. $\mathcal{C}^\mu_{\text{fin}}$ has the amalgamation property for strong extensions.

Proof. Consider $B \leq M$ and $B \leq N$ in $\mathcal{C}^\mu_{\text{fin}}$ and assume $M \otimes_B N$ does not belong to $\mathcal{C}^\mu_{\text{fin}}$. We may assume that N is a minimal extension of B (otherwise we could build a finite chain of minimal extensions). Since $M \otimes_B N \notin \mathcal{C}^\mu_{\text{fin}}$, it contains a long pseudo Morley sequence (A_i) of some α over B'. Now by Lemma 1.5 there are two cases:

1. $B' \subseteq M$. Since $M \in \mathcal{C}^\mu_{\text{fin}}$, there is an A_i which lies not completely in M.
   $(A_i \cap M) \cup B'$ is a closed subset of $A_i \cup B'$, but by minimality this can only be if $A_i \cap M$ is empty. Hence $A_i \subseteq N \setminus M = N \setminus B =: A$.
   Now $B \leq N = B \cup N \setminus B = B \cup A$ is minimal, implying the pair A/B is minimal. All elements in M which are connected to A (resp. A_i) lie in B (free amalgam). Hence by the minimality of A/B , also A/M is minimal. Now

$$\begin{aligned} 0 &\overset{M \text{ strong}}{\leq} \delta(A_i/M) = \delta(A_i \cup M) - \delta(M) = \delta(A_i \cup B' \cup M) - \delta(M) \\ &\leq \delta(A_i \cup B') + \delta(M) - \delta((A_i \cup B') \cap M) - \delta(M) = \delta(A_i \cup B') - \delta(B') = \delta(A_i/B') = 0 \end{aligned}$$

   and hence $\delta(A_i/M) = 0$. But by lemma 0.14 this can only be if $A_i = A$.

   Furthermore, A/B' is a good pair and by the definition of the free amalgam all elements of N, which are conntected to $A = N \setminus B$, must lie in B. Hence $B' \subseteq B$. Again, since $N \in \mathcal{C}^\mu_{\text{fin}}$, there is an A_j which lies in $M \setminus B$. Now B' is the basis of A/B and A_j/B, so they must be isomorphic. Then we can embed $N = B \cup A$ in M by mapping it onto $B \cup A_j$ and we have found an amalgamation.

2. $A_i \subseteq N \setminus M$ for some i. Since A_i/B' is minimal, by corollary 0.15 $B' \subseteq N$. N belongs to $\mathcal{C}^\mu_{\text{fin}}$, so some A_j lies in $M \setminus B$. As above $B' \subseteq B$ and we proceed as in the first case.

□

Definition 1.9. We define M^μ to be the Fraïssé limit of $\mathcal{C}^\mu$.

The following will be a complete axiomatisation of the theory of M^μ:

Definition 1.10. M is a model of T^μ if the following conditions hold:

a) M belongs to $\mathcal{C}^\mu$.

b) No prealgebraic minimal extension of M belongs to $\mathcal{C}^\mu$.

c) M is infinite.

Lemma 1.11. T^μ is ∀∃-axiomatisable.

Proof. Clearly, condition c) can be elementarily expressed by ∀∃-sentences.
For the other conditions notice first, the (non-)existence of a long pseudo Morley sequence for a given isomorphism type α can be expressed by a ∀∃-sentence. (The isomorphism type is completely discribed by the relations which hold and this can be easily expressed. Then its again easy to express that there exits $\leq \mu(\alpha)$ or $> \mu(\alpha)$ disjoint sets which also hold the required relations.)
By using such a sentence for every isomorphism type α in M, we get a set of axioms to express condition a).

Now for condition b): We show: If we have $M \in \mathcal{C}^\mu$ and A/M a prealgebraic minimal pair with basis B and α the isomorphism type for A/B, then there is only a finite number of codes α' which can have a long pseudo Morley sequence in $N = M \cup A = M \otimes (B \cup A)$. If we express the existence of a long Morley for each of the α' as a ∀∃-sentence, then the disjunction expresses the existence of any long Morley sequence and is

also a ∀∃-sentence.
So let $M \in \mathcal{C}^\mu$ and A/M a prealgebraic minimal pair with basis B and α the isomorphism type for A/B and assume (A'_i) is a long pseudo Morley sequence of α' in over B' in N. By the Mainlemma 1.5 there is two cases:

1.) $B' \subseteq M$. As in the proof of the amalgamation property we conclude that some A'_i equals A and that $B' \subseteq B$. Now $B' = B$, since $A'_i/B' = A/B'$ and A/B are both good. Then we have $\alpha' = \alpha$, so only one possible isomorphism type.

2.) Some A'_i lies in A. The size of B' can be bounded by $|B'| \leq 2 \cdot |A'_i| \leq 2 \cdot |A|$. So there is only finitely many possibilities for α'.

□

Corollary 1.12. T^μ is model complete.

Proof. We will see later, that T^μ is strongly minimal and strongly minimal theories in a countable language are uncountably categorical. Now use Lindströms theorem: A ∀∃-theory which is categorical in some cardinal is model complete. □

Recall 1.13. M is rich (regarding $\mathcal{C}^\mu$) if:
If B is closed in M and $B \leq C \in \mathcal{C}^\mu_{\text{fin}}$, then C can be strongly embedded in M over B.

Proposition 1.14. A structure M is rich iff. it is an ω-saturated model of T^μ.

As in the case of T^0 we can follow:

Corollary 1.15. T^μ axiomatises the complete theory of M^μ.

For the proof of Proposition 1.14 we need the following lemma:

Lemma 1.16. In every ω-saturated structure $M \in \mathcal{C}^\mu$, the algebraic closure contains the geometric closure.

Proof. For any finite set X the property $X \leq M$ is equiv to $\delta(D/D \cap X) \geq 0$ for all $D \subseteq M$, which can be expressed by sentences. Hence $\text{cl}(X)$ is algebraic over X. Therefore we might assume $X \leq M$, when proving $\text{Cl}(X)$ is algebraic over X. Then by lemma0.20 $\text{Cl}(X)$ is the union of all extensions C with $\delta(C/X) = 0$.
So it suffices to show, that every prealgebraic minimal extension A/X is algebraic. Let B be the basis of A/X and α the type of A/B. Then any sequence of (A_i) of realisations of $\text{tp}(A/X)$ is a pseudo Morley sequence of α. Hence there can only be finitely ($\leq \mu(\alpha)$) many and by ω-saturation this implies A is algebraic over X. □

Proof of Prop 1.14. "$\Leftarrow$ "Let M be an ω-saturated model of T^μ. To show, that M is rich, consider $B \leq M$ and an extension $B \leq C \in \mathcal{C}^\mu_{\text{fin}}$. We may assume that the extension is minimal (o/w we can consider a finite chain of minimal extensions). By lemma0.16 there are two cases:

1. $\delta(C/B) = 0$. Then $M \otimes_B C$ would be a prealgebraic minimal extension of M, hence $M \otimes_B C \notin \mathcal{C}^\mu$. In this case, C embeds over B into M as in the proof of Lemma1.8.
2. $C = B \cup \{c\}$ with $\delta(c/B) = 1$. Then c is not connected to B. To embed C strongly into M we take c' outside of $\text{Cl}(B)$. Such a c' exists, since ω-saturation implies that $\text{acl}(B)$ is a proper subset of the infinite structure M. Now $B \cup \{c'\}$ is strong, since for any $B \cup \{c'\} \subseteq A \subseteq M$ we get

$$\begin{aligned} 1 = d(c'/B) &= \min\{\delta(A') | B \cup \{c'\} \subseteq A'\} - \delta(\text{cl}(B)) \leq \delta(A) - \delta(B) \\ &= \delta(A) - |B| + |R(B)| \leq \delta(A) + 1 - |B \cup \{c'\}| + |R(B \cup \{c'\})| = \delta(A) - \delta(B \cup \{c') + 1 \end{aligned}$$

or equiv. $\delta(A) \geq \delta(B \cup \{c'\})$.

"$\Rightarrow$"Assume M is rich. Now condition a) automatically holds and for c) we notice, that all F_n belong to $\mathcal{C}^\mu$. Hence M^μ is infinite.
For the second one let A/M be prealgebraic minimal extension with basis B and α the type of A/B. Assume that $M \cup A$ belongs to $\mathcal{C}^\mu$. Now take any finite extension C_0 of B, which is closed in M. Since $C_0 \leq M$ and $C_0 \leq C_0 \cup A \in \mathcal{C}^\mu_{\text{fin}}$, by richness M contains a copy A_0 of A over C_0 (by construction A_0 is disjoint from C_0). We now choose $C_1 \leq M$, which contains $C_0 \cup A_0$ and construct as before a copy A_1 of A over C_1, which is disjoint from C_1, so in particular A_1 is disjoint from A_0. By continuing, we construct an infinite pseudo Morley sequence (A_i) of α. However this is a contradiction to $M \in \mathcal{C}^\mu$.
ω-saturation follows again as in the T^0-case from the other direction. (If M' is any ω-saturated model of T^μ. Then M' is rich and therefore partially isomorphic to any rich M and this implies that also M is ω-saturated.) $\square$

Lemma 1.17. Let M_1 and M_2 be two models of T^μ. Then $\overline{a}_1 \in M_1$ and $\overline{a}_2 \in M_2$ have the same type iff $\overline{a}_1 \mapsto \overline{a}_2$ extends to an isomorphism $\mathrm{cl}(\overline{a}_1) \to \mathrm{cl}(\overline{a}_2)$.

The proof proceeds exactly as for T^0 and will be ommitted here.

Theorem 1.18. T^μ is strongly minimal.

Proof. We show, that there is only one non-algebraic type $\mathrm{tp}(c/B)$. There is two different cases $d(c/B) = 0$ and $d(c/B) = 1$.
If $d(c/B) = 0$, then $c \in \mathrm{Cl}(B) \subseteq \mathrm{acl}(B)$, hence $\mathrm{tp}(c/B)$ is algebraic.
Now by lemma0.21 in the case $d(c/B) = 1$ we know c is not connected to $\mathrm{cl}(B)$ and $\mathrm{cl}(B) \cup \{c\}$ is closed. By the construction of M^μ via the Fraïssé limit, if we have another {c'} with those properties we can expand any map fixing $\mathrm{cl}(B)$ and sending c to c' to an automorphism. Then Lemma1.17 yields that $\mathrm{tp}(c/B) = \mathrm{tp}(c'/B)$. $\square$

Corollary 1.19. In any model of T^μ the geometric closure equals the algebraic closure. In particular, the relative dimension $d(A/B)$ is the Morley rank of $\mathrm{tp}(A/B)$.

Proof. By lemma 1.16 we already know that the geometric closure is contained in the algebraic one. Now the he proof of theorem1.18 gives us

$$c \notin \mathrm{Cl}(B) \Leftrightarrow d(c/B) = 1 \Rightarrow c \notin \mathrm{acl}(B).$$

$\square$

Proposition 1.20. T^μ is not locally modular.

Proof. This follows as in the T^0 case, because by the examples in 1.7 $C_{nm} \in \mathcal{C}^\mu$ and $C_{nm} \cup \{d\} \in \mathcal{C}^\mu$ for some d not connected to C_{nm}. $\square$

Remark 1.21. For any $n < \omega$ we consider $M_n = \{x_1, ..., x_n\}$ with the relations $R(M_n) = \{\{x_{i-1}, x_i, x_{i+1}| 1 \leq i \leq n (x_{-1} = x_n, x_{n+1} = x_1\}$. We can check that $M_n \in \mathcal{C}^\mu$ and $d(M_n) = 0$. Therefore we may assume $M_n \leq M^\mu$ and $M_n \subseteq \mathrm{acl}(\emptyset)$. Hence, $\mathrm{acl}(\emptyset)$ is infinite and together with strongly minimal, this yields that T^μ has weak elimination of imaginaries.
Because of this we can consider acl whenever we need acl^{eq}.

Lemma 1.22. Suppose $A, B \leq M^\mu$ and $C = A \cap B$. Then $A \mathop{\smile}\limits^{\mid} B|C$ iff $A \cup B = A \otimes_C B \leq M^\mu$.

Proof. First assume A, B are finite. We know $A \mathop{\smile}\limits^{\mid} B|C$ iff $d(A/B) = d(A/C)$. If this holds, we can use, that $A, B, A \cap B = C$ are closed and get

$$\delta(AB) \geq d(AB) = d(A) + d(B) - d(C) = \delta(A) + \delta(B) - \delta(C) \overset{\text{submod.}}{\geq} \delta(AB).$$

$d(AB) = \delta(AB)$ implies, that $A \cup B \leq M$. Furthermore, $\delta(A) + \delta(B) - \delta(C) = \delta(AB)$ implies $|R(A \cup B)| = |R(A)| + |R(B)| - |R(C)|$ and hence $R(A \cup B) = R(A) \cup R(B)$. Finally we get $A \cup B = A \otimes_C B$.
On the other hand, if $A \cup B = A \otimes_C B \leq M^\mu$, the we get $d(AB) \overset{\text{closed}}{=} \delta(AB) \overset{A \otimes_C B}{=} \delta(A) + \delta(B) - \delta(C) \overset{\text{closed}}{=} d(A) + d(B) - d(C)$ or equivalently $d(A/B) = d(A/C)$.

If A, B are not finite. We use the finite character of independence:

$$A \downarrow B|C \text{ iff } A' \downarrow B'|C \text{ for every finite } A' \leq A, B' \leq B$$

and that by lemma 0.12 and lemma 0.23 we know

$$A \cup B = A \otimes_C B \leq M \text{ iff } A' \cup B' = A' \otimes_C B' \leq M \text{ for every finite } A' \leq A, B' \leq B.$$

□

Definition 1.23. A δ-function f is called flat on $E_1, ..., E_n$, if:

$$\sum_{\Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} f(E_\Delta) \leq 0,$$

where $E_{Delta} = \bigcap_{i \in \Delta} E_i$ and $E_\emptyset = \bigcup_{1 \leq i \leq n} E_i$.

Proposition 1.24. In structures from $\mathcal{C}^0$, d is flat on Cl-closed finite-dimensional sets.

For the proof we need the following lemma:

Lemma 1.25. If $E_1, ..., E_n$ are Cl-closed finite-dimensional sets, we can choose finite closed sets $A_i \leq E_i$ such that $\mathrm{Cl}(A_\Delta) = \mathrm{Cl}(\bigcap_{i \in \Delta} A_i) = E_\Delta$ for all $\Delta \neq \emptyset$ and $\mathrm{Cl}(A_\emptyset) = \mathrm{Cl}(E_\emptyset)$, where $A_\emptyset := \mathrm{cl}(A_1 \cup ... \cup A_n)$.

Proof. For every nonempty $\Delta \subseteq \{1, ..., n\}$, pick a finite F_Δ such that $\mathrm{Cl}(F_\Delta) = E_\Delta$. Let $A_i \subseteq E_i$ be a finite closed subset and $\bigcup\{F_\Delta | i \in \Delta \subseteq \{1, ..., n\}\}$. Then for any nonempty Δ we have:

$$E_\Delta = \mathrm{Cl}(F_\Delta) \overset{F_\Delta \subseteq A_\Delta}{\subseteq} \mathrm{Cl}(A_\Delta) \overset{A_\Delta \subseteq E_\Delta}{\subseteq} \mathrm{Cl}(E_\Delta) = E_\Delta.$$

Also,

$$E_\emptyset = E_1 \cup ... \cup E_n = \mathrm{Cl}(A_1) \cup ... \cup \mathrm{Cl}(A_n) \subseteq \mathrm{Cl}(A_1 \cup ... \cup A_n) \subseteq \mathrm{Cl}(\mathrm{cl}(A_1 \cup ... \cup A_n)) = \mathrm{Cl}(A_\emptyset)$$

and

$$A_\emptyset = \mathrm{cl}(A_1 \cup ... \cup A_n) = \mathrm{cl}(\mathrm{Cl}(E_1) \cup ... \cup \mathrm{Cl}(E_n)) \subseteq \mathrm{cl}(\mathrm{Cl}(E_1 \cup ... \cup E_n)) = \mathrm{Cl}(E_1 \cup ... \cup E_n) = \mathrm{Cl}(E_\emptyset).$$

□

Proof of Proposition 1.24. Let $E_1, ..., E_n$ be Cl-closed finite-dimensional sets and choose finite $A_i \leq E_i$ as in lemma 1.25. Then we have $d(E_\Delta) = d(A_\Delta) = \delta(A_\Delta) = |A_\Delta| - |R(A_\Delta)|$. From an inclusion-exclusion argument we know $\sum_{\Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} |A_\Delta|$ and

$$|R(A_1) \cup ... \cup R(A_n)| = - \sum_{\emptyset \neq \Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} |\bigcap_{i \in \Delta} R(A_i)| = - \sum_{\emptyset \neq \Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} |R(A_\Delta)|.$$

Together this yields

$$\begin{aligned} \sum_{\Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} d(E_\Delta) &= \sum_{\Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} |A_\Delta| - (\sum_{\Delta \subseteq \{1,...,n\}} (-1)^{|\Delta|} |R(A_\Delta)|) \\ &= |R(A_1) \cup ... \cup R(A_n)| - |R(A_\emptyset)| = |R(A_1) \cup ... \cup R(A_n)| - |R(A_1 \cup ... \cup A_n)| \leq 0 \end{aligned}$$

□

Proposition 1.26. There is no infinite group interpretable in T^μ.

Proof. Let G be a groupt interpreted in a model M of T^μ, i.e. definable in M^{eq}. We assume first, that G is 0-definable in M. Let g be the Morley rank of G and pick three independent elements a_1, a_2, a_3 of Morley rank g. Now we define $b_1 := a_1 \cdot a_2$, $b_3 := a_2 \cdot a_2 \cdot a_3$ and $b_2 := b_1 \cdot a_3 = a_1 \cdot b_3$. Furthermore, let $L_1 := \{a_1, a_2, b_1\}$, $L_2 := \{a_2, a_3, b_3\}$, $L_3 := \{a_1, b_2, b_3\}$ and $L_4 := \{a_3, b_1, b_2\}$. By definition of the L_i one element in L_i is always the product of the other two, hence each element is algebraic over the other two. This also implies that $g \overset{b_1 \in G}{\geq} d(b_i) \geq d(b_1/a_1) = d(a_2/a_1) = g$ and that three elements, which do not all lie in one L_i are independent. For example,

$$d(a_1, b_1, b_3) \overset{a_2, b_2 \in \mathrm{acl}(a_1, b_1, b_3)}{=} d(a_1, b_1, a_2, b_3, b_2) \overset{\substack{b_1 \in \mathrm{acl}(a_1, a_2) \\ a_3 \in \mathrm{acl}(a_2, b_3)}}{=} d(a_1, a_2, b_2, b_3, a_3)$$
$$\overset{b_2, b_3 \in \mathrm{acl}(a_1, a_2, a_3)}{=} d(a_1, a_2, a_3) = 3g = d(a_1) + d(b_1) + d(b_3).$$

Now we define $E_i = \mathrm{Cl}(L_i)$. Now if L_i, L_j intersect in x, then $E_i \downarrow E_j | \mathrm{Cl}(x)$ and hence $E_{ij} = E_i \cap E_j = \mathrm{Cl}(x)$. The intersection of L_i, L_j, L_k is empty and hence $E_{ijk} = E_i \cap E_j \cap E_k = \mathrm{Cl}(\emptyset)$. We get:

- $d(E_\emptyset) = d(E_1 \cup E_2 \cup E_3 \cup E_4) = 3g$
- $d(E_i) = 2g$
- $d(E_{ij}) = g$
- $d(E_{ijk}) = d(E_{ijkl}) = \mathrm{Cl}(\emptyset) = 0.$

Now the flatness yields $0 \geq 3g - 4 \cdot (2g) + 6g = g$. Hence, G is finite.

Assume G is definable in M^{eq} with parameters $A \subseteq M$. Since T^μ has weak elimination of imaginaries, we can replace the group diagram of G by a group diagram in M with the same Morley rank over A. Now we are back in the case from above. ☐

1.1 CM-triviality

In this section we will show, that M^μ is weakly CM-trivial, which is equivalent to being not 2-ample. We will see in Thomas Kochs talk any structure, which interprets a field is n-ample for all n. Therefore we can conclude that M^μ does not interpret any infinite field, which is the final contradiction to Zilbers conjecture.

Definition 1.27. A stable structure M is CM-trivial, if the following holds: Let C, A, B be algebraically closed. Assume $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = A$. Then $\mathrm{Cb}(C/A) \subseteq \mathrm{Cb}(C/A \cup B)$.

Proposition 1.28. For a stable structure M the following condition are equivalent:

(CMT1) Suppose B_1, B_2 are independent over $E = \mathrm{acl}(E)$ and $\mathrm{acl}(B_1, B_2) \cap \mathrm{acl}(E, B_i) = B_i$, and $B_i \cap E = A$. Then B_1, B_2 are independent over A.

(CMT2) If E is algebraicaly closed, $C_1 \downarrow C_2 | E$, then $C_1 \downarrow C_2 | (\mathrm{acl}(C_1, C_2) \cap E)$.

(CMT3) Let C, A, B be algebraically closed and $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = A$. Then $\mathrm{Cb}(C/A) \subseteq \mathrm{acl}(\mathrm{Cb}(C/A \cup B))$.

Remark 1.29. We will see in a later talk, that (CMT3) is equivalent to being CM-trivial.

proof of Proposition 1.28. (1) ⇒ (2): Let C_1, C_2, E be as in (CMT2). Let $B_i := \mathrm{acl}(C_1, C_2) \cap \mathrm{acl}(E, C_i)$. Then $C_i \subseteq B_i \underset{=}{\subseteq} \mathrm{acl}(C_1, C_2)$, so $\mathrm{acl}(B_1, B_2) = \mathrm{acl}(C_1, C_2)$ and also $C_i \subseteq B_i \subseteq \mathrm{acl}(E, C_i)$, so $\mathrm{acl}(E, B_i) = \mathrm{acl}(E, C_i)$. Thus B_i def. $\mathrm{acl}(C_1, C_2) \cap \mathrm{acl}(E, C_i) = \mathrm{acl}(B_1, B_2) \cap \mathrm{acl}(E, B_i)$. Also $B_i \cap E = \mathrm{acl}(C_1, C_2) \cap \mathrm{acl}(E, C_i) \cap E \overset{E \subseteq \mathrm{acl}(E, C_i)}{=} \mathrm{acl}(C_1, C_2) \cap E =: A$. Hence we can apply (CMT1) and get $B_1 \downarrow B_2 | A$. In particular, since $C_i \subseteq B_i$ we have $C_1 \downarrow C_2 | A$ (CMT2).

(2) ⇒ (3): Let C, A, B be algebraically closed, $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = A$ as in the definition of CM-trivial. We first assume $A \subseteq B$. Let $Y := \mathrm{acl}(\mathrm{Cb}(C/B))$, so by a corollary 0.1 we know $C \downarrow B | Y$ and in particular $C \downarrow A | Y$. By (CMT2), $C \downarrow A | (Y \cap \mathrm{acl}(C \cup A))$. Now $Y \cap A \overset{\text{assump.}}{=} Y \cap \mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = Y \cap \mathrm{acl}(A \cup C) \cap B \overset{Y \subseteq B}{=} Y \cap \mathrm{acl}(C \cup A)$. Thus by cor0.1 we have $C \downarrow A | Y \cap A$ and therefore $\mathrm{Cb}(C/A) \subseteq Y$. ☐

Lemma 1.30. M^μ is CM-trivial.

Proof. We will show (CMT1), so suppose B_1, B_2 are independent over $E = \operatorname{acl}(E)$ with $\operatorname{acl}(B_1, B_2) \cap \operatorname{acl}(E, B_i) = B_i$ and $B_i \cap E = A$.
Define $\overline{B}_i := \operatorname{acl}(B_i \cup E)$. The independence of B_1, B_2 over E also implies that $B_1 \downarrow B_2 | E$ and by lemma 1.22 $\overline{B}_1 \cup \overline{B}_2 = \overline{B}_1 \otimes_E \overline{B}_2 \leq M^\mu$. Now by assumption $\operatorname{acl}(B_1 \cup B_2) \cap \overline{B}_i = B_i$ and hence $\operatorname{acl}(B_1 \cup B_2) \cap \operatorname{acl}(\overline{B}_1 \cup \overline{B}_2) = B_1 \cup B_2$. This implies $B_1 \cup B_2 \leq \overline{B}_1 \cup \overline{B}_2 \leq M^\mu$.
Because of independence we get

$$B_i \cap E \overset{E \subseteq B_1 \cap B_2}{\subseteq} B_1 \cap B_2 \subseteq B_i \cap (\overline{B}_1 \cap \overline{B}_2) \overset{E = \overline{B} \cap \overline{B}_2}{=} B_i \cap E = A$$

and using $\overline{B}_1 \cup \overline{B}_2 = \overline{B}_1 \otimes_E \overline{B}_2$ this implies $\overline{B}_1 \otimes_A \overline{B}_2$. And by Lemma 1.22 we get $B_1 \downarrow B_2 | A$. □