On ampleness and pseudo-Anosov homeomorphisms in the free group

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Abstract: We use pseudo-Anosov homeomorphisms of surfaces in order to prove that the first-order theory of non-Abelian free groups, T_{fg}, is n-ample for any $n \in \omega$. This result adds to the work of Pillay, which proved that T_{fg} is non-CM-trivial. The sequence witnessing ampleness is a sequence of primitive elements in $\mathbb{F}_\omega$.

Our result provides an alternative proof to the main result of a recent preprint by Ould Houcine and Tent.

Key words: Free groups, pseudo-Anosov homeomorphisms, geometric stability theory

1. Introduction

The notion of n-ampleness, for some natural number n, fits in the general context of geometric stability theory. As the definition may look artificial or technical, we first give the historical background of its development. We start by working in a vector space V and we consider 2 finite dimensional subspaces $V_1, V_2 \leq V$. Then one can see that $dim(V_1 + V_2) = dim(V_1) + dim(V_2) - dim(V_1 \cap V_2)$, and the point really is that V_1 is linearly independent from V_2 over $V_1 \cap V_2$. In an abstract stable theory the notion of linear independence is replaced by forking independence (see Section 2) and the above property gives rise to the notion of 1-basedness. A stable theory T is *1-based* if there are no a, b such that $acl^{eq}(a) \cap acl^{eq}(b) = acl^{eq}(\emptyset)$ and a forks with b over $\emptyset$. The notion of 1-basedness turned out to be very fruitful in model theory and one of the major results concerning this notion was the following theorem by Hrushovski and Pillay [7].

Theorem 1.1 *Let $\mathcal{G}$ be a 1-based stable group. Then every definable set $X \subseteq G^n$ is a Boolean combination of cosets of almost $\emptyset$-definable subgroups of G^n. Moreover, G is Abelian-by-finite.*

On the other hand, Hrushovski's seminal work in refuting Zilber's trichotomy conjecture (see [6]) produced "new" strongly minimal sets that had an interesting property. Hrushovski isolated this property and called it CM-triviality (for Cohen–Macaulay). A stable theory T is CM-*trivial* if there are no a, b, c such that a forks with c over $\emptyset$, a is independent from c over b, $acl^{eq}(a) \cap acl^{eq}(b) = acl^{eq}(\emptyset)$, and finally $acl^{eq}(a, b) \cap acl^{eq}(a, c) = acl^{eq}(a)$. A kind of an analogue to the "moreover statement of the above theorem was proven by Pillay in [20].

Theorem 1.2 *A CM-trivial group of finite Morley rank is nilpotent-by-finite.*

Pillay first realized the pattern and proposed a hierarchy of ampleness, non-1-basedness (1-ampleness) and non-CM-triviality (2-ampleness) being the first 2 items in it (see [22]). His definition needed some "fine tuning" as observed by Evans [2].

Definition 1.3 ([2]) *Let T be a stable theory and $n \geq 1$. Then T is n-ample if (after possibly adding some parameters) there are $a_0, a_1, \ldots, a_n$ such that:*

1. *a_0 forks with a_n over $\emptyset$;*

2. *a_{i+1} does not fork with $a_0, \ldots, a_{i-1}$ over a_i, for $1 \leq i < n$;*

3. *$acl^{eq}(a_0) \cap acl^{eq}(a_1) = acl^{eq}(\emptyset)$;*

4. *$acl^{eq}(a_0, \ldots, a_{i-1}, a_i) \cap acl^{eq}(a_0, \ldots, a_{i-1}, a_{i+1}) = acl^{eq}(a_0, \ldots, a_{i-1})$, for $1 \leq i < n$.*

For the precise definitions of the above-mentioned model's theoretic notions we refer the reader to Section 2.

In this paper we give a sequence of primitive elements of $\mathbb{F}_\omega$ witnessing n-ampleness, for any $n < \omega$.

The paper is structured as follows. The next section serves as an introduction to the notions (of both model theory and geometric group theory) needed in order to place the result into context. The main purpose is to make our exposition as friendly as possible to the general reader, while we still give references when we feel that this is not possible.

In the third section we give a sequence of primitive elements and prove that this sequence witnesses n-ampleness for any $n < \omega$. Actually, our methods provide an "abundance" of examples to ampleness.

Finally, we add an appendix, in which we record some strengthenings and alternative transparent proofs of Theorems 2.1, 2.2, and 2.3 from [32], which prove that the "basic" imaginaries (see Definition 2.1) cannot be eliminated.

Remark 1.4 *The idea for the sequence witnessing ampleness (see Section 3) came to us after reading [16], in which the main result is the ampleness of the theory of non-Abelian free groups (or more generally the theory of any (noncyclic) torsion-free hyperbolic group). We first posted our sequence in [34], where we used results from [16] in order to prove that our sequence satisfies the algebraic criteria (3) and (4) of Definition 1.3.*

Conceptually, the 2 sequences are very different and the main advantage of our sequence is that it reduces the work of satisfying the algebraic conditions to a (well-absorbed) fact about homeomorphisms of surfaces. Thus, in this paper, we give an alternative proof to the one given in [34] with the hope that it will add to the understanding of the first-order theory of non-Abelian free groups.

2. Preliminaries

In this section we collect some basic definitions and facts about model theory and geometric group theory. To be more precise, in the first subsection we will define the notion of imaginaries and explain various notions of elimination of imaginaries. In the second subsection, we explain the notion of forking independence and we connect it with the notion of generic types in stable groups. In the next section, we specialize these notions to the first-order theory of the free group. In the fourth subsection, we will define amalgamated free products and give some normal form theorems for elements in an amalgamated free product. In the last subsection, we will define pseudo-Anosov homeomorphisms of surfaces and record some useful facts about them.

The reader should note that our treatment is by no means complete but we will always provide references for notions and results that are not adequately explained.

2.1. Imaginaries

We fix a first-order structure $\mathcal{M}$ and we are interested in the collection of definable sets in $\mathcal{M}$, i.e. all subsets of some Cartesian power of $\mathcal{M}$, which are the solution sets of first-order formulas (in $\mathcal{M}$). In some cases one can easily describe this collection, usually thanks to some quantifier elimination result. For example, as algebraically closed fields admit (full) quantifier elimination (in the language of rings), the class of definable sets coincides with the class of constructible sets, i.e. the class consisting of Boolean combinations of Zariski closed sets. On the other hand, although free groups admit quantifier elimination down to Boolean combinations of $\forall\exists$ formulas (see [28, 29]), the "basic" definable sets are not so easy to describe.

Suppose X is a definable set in $\mathcal{M}$. One might ask whether there is a canonical way to define X, i.e. is there a tuple $\bar{b}$ and a formula $\psi(\bar{x}, \bar{y})$ such that $\psi(\mathcal{M}, \bar{b}) = X$ but for any other $\bar{b}' \neq \bar{b}$, $\psi(\mathcal{M}, \bar{b}') \neq X$?

To give a positive answer to the above-mentioned question, one has to move to a "mild" expansion of $\mathcal{M}$ called $\mathcal{M}^{eq}$. Very briefly, $\mathcal{M}^{eq}$ is constructed from $\mathcal{M}$ by adding a new sort for each $\emptyset$-definable equivalence relation, $E(\bar{x}, \bar{y})$, together with a class function $f_E : M^n \to M_E$, where M_E (the domain of the new sort corresponding to E) is the set of all E-equivalence classes. The elements in these new sorts are called *imaginaries*. In $\mathcal{M}^{eq}$, it is not hard to see that one can assign to each definable set a canonical parameter in the sense discussed above. Indeed, let X be the solution set of the formula $\phi(\bar{x}, \bar{b})$ in $\mathcal{M}$ and consider the equivalence relation $E(\bar{y}_1, \bar{y}_2) := \forall \bar{x}(\phi(\bar{x}, \bar{y}_1) \leftrightarrow \phi(\bar{x}, \bar{y}_2))$. The element $f_E(\bar{b})$ in $\mathcal{M}^{eq}$ then serves as the "canonical parameter" such that when it is plugged into the formula $\psi(\bar{x}, z_E) := \exists \bar{y}(\phi(\bar{x}, \bar{y}) \wedge f_E(\bar{y}) = z_E)$, where z_E denotes a variable that takes values in the E-sort, it defines canonically the set X.

An element a of $\mathcal{M}^{eq}$ is *algebraic* (respectively *definable*) over $A \subseteq \mathcal{M}^{eq}$, denoted $a \in acl^{eq}(A)$ (respectively $a \in dcl^{eq}(A)$), if there exists a first-order formula over A with finitely many solutions (respectively exactly one solution) in $\mathcal{M}^{eq}$ containing a.

We say that $\mathcal{M}$ *eliminates imaginaries* if there is a saturated elementary extension of $\mathcal{M}$ in which all definable sets can be assigned a canonical parameter. Equivalently, $\mathcal{M}$ eliminates imaginaries if it has a saturated elementary extension $\mathbb{M}$ in which for any element $\mathfrak{e}$ of $\mathbb{M}^{eq}$, there is a finite tuple $\bar{b} \in \mathbb{M}$ such that $\mathfrak{e} \in dcl^{eq}(\bar{b})$ and $\bar{b} \in dcl^{eq}(\mathfrak{e})$.

One can alter the above definition to obtain the following weaker notions of elimination of imaginaries. We say that $\mathcal{M}$ *weakly* (respectively *geometrically*) *eliminates imaginaries* if it has a saturated elementary extension $\mathbb{M}$ in which for any element $\mathfrak{e}$ of $\mathbb{M}^{eq}$, there is a finite tuple $\bar{b} \in \mathbb{M}$ such that $\mathfrak{e} \in dcl^{eq}(\bar{b})$ (respectively $\mathfrak{e} \in acl^{eq}(\bar{b})$) and $\bar{b} \in acl^{eq}(\mathfrak{e})$.

The interested reader can find more details in [25, Sections 16.4 and 16.5].

2.2. Forking independence

For a quick introduction to forking independence in stable theories, we refer the reader to [9, Section 2] or [19, Section 2.2], and more thorough references are [21] and [25].

A first-order theory T is called stable if it supports a notion of independence (between tuples in an "enough" saturated model of T) satisfying certain properties. As a matter of fact, in a stable theory, there is exactly one notion of independence with the desired properties, which is called forking independence.

We work in a "big" saturated model, $\mathbb{M}$, of a stable theory T (what model theorists often call the *monster model* see; [12, p. 218]). Let $\bar{a}, \bar{b}, \ldots$ denote finite tuples in $\mathbb{M}$ and $A, B, \ldots$ a small subset of $\mathbb{M}$, i.e.

$|A|, |B| < |\mathbb{M}|$. We say that $\bar{a}$ *forks with* (is *not independent from*) B over A if there is $\phi(\bar{x}, \bar{b}) \in tp(\bar{a}/B)$ and an indiscernible sequence $(\bar{b}_i)_{i<\omega}$ over A with $tp(\bar{b}/A) = tp(\bar{b}_i/A)$, such that $\{\phi(\bar{x}, \bar{b}_i) \mid i < \omega\}$ is inconsistent.

Moreover, a sequence of tuples $\bar{a}_1, \ldots, \bar{a}_k$ is called an *independent set (over A)* if $\bar{a}_i$ does not fork with $\bar{a}_1, \ldots, \bar{a}_{i-1}$ over A, for all $i \leq k$.

A group, G, is called stable if it is definable in some model of a stable theory T. An important aspect of stable groups is the existence of generic types. A type $tp(a/A)$ with $a \in G$ is called *generic* if whenever $g \in G$ and a does not fork with g over A, then $g \cdot a$ does not fork with g over A. Moreover, an element $a \in G$ is called *generic* (over A) if $tp(a/A)$ is generic.

The development of stable group theory in full generality is mostly due to Poizat; an elegant reference for a more thorough reading as well as for motivation of the above definitions is [27].

2.3. The free group

We now specialize all the above model theoretic notions to the first-order theory of the free group (considered in the natural language for groups, i.e. $(\cdot, {}^{-1}, 1)$).

We start by defining some "basic" families of imaginaries.

Definition 2.1 *Let $\mathbb{F}$ be a non-Abelian free group. The following equivalence relations in $\mathbb{F}$ are called basic.*

(i) $E_1(a, b)$ *if and only if there is $g \in \mathbb{F}$ such that $a^g = b$. (conjugation)*

$(ii)_m$ $E_{2_m}((a_1, b_1), (a_2, b_2))$ *if and only if either $b_1 = b_2 = 1$ or $b_1 \neq 1$ and $C_{\mathbb{F}}(b_1) = C_{\mathbb{F}}(b_2) = \langle b \rangle$ and $a_1^{-1} a_2 \in \langle b^m \rangle$. (m-left-coset)*

$(iii)_m$ $E_{3_m}((a_1, b_1), (a_2, b_2))$ *if and only if either $b_1 = b_2 = 1$ or $b_1 \neq 1$ and $C_{\mathbb{F}}(b_1) = C_{\mathbb{F}}(b_2) = \langle b \rangle$ and $a_1 a_2^{-1} \in \langle b^m \rangle$. (m-right-coset)*

$(iv)_{m,n}$ $E_{4_{m,n}}((a_1, b_1, c_1), (a_2, b_2, c_2))$ *if and only if either $a_1 = a_2 = 1$ or $c_1 = c_2 = 1$ or $a_1, c_1 \neq 1$ and $C_{\mathbb{F}}(a_1) = C_{\mathbb{F}}(a_2) = \langle a \rangle$ and $C_{\mathbb{F}}(c_1) = C_{\mathbb{F}}(c_2) = \langle c \rangle$ and there is $\gamma \in \langle a^m \rangle$ and $\epsilon \in \langle c^n \rangle$ such that $\gamma b_1 \epsilon = b_2$. (m, n-double-coset)*

It is almost immediate that m-left cosets eliminate m-right cosets (and vice versa), so from now on we are economic and forget about the m-right-cosets.

Sela proved the following theorem concerning imaginaries in non-Abelian free groups (see [32, Theorem 4.4]).

Theorem 2.2 *Let $\mathbb{F}$ be a non-Abelian free group. Let $E(\bar{x}, \bar{y})$ be a definable equivalence relation in $\mathbb{F}$, with $|\bar{x}| = m$. Then there exist $k, l < \omega$ and a definable relation*

$$R_E \subseteq \mathbb{F}^m \times \mathbb{F}^k \times S_1(\mathbb{F}) \times \ldots \times S_l(\mathbb{F})$$

such that:

(i) *each $S_i(\mathbb{F})$ is one of the basic sorts;*

(ii) *for each $\bar{a} \in \mathbb{F}^m$, $|R_E(\bar{a}, \bar{z})|$ is uniformly bounded (i.e. the bound does not depend on $\bar{a}$);*

(iii) $\forall \bar{z}(R_E(\bar{a}, \bar{z}) \leftrightarrow R_E(\bar{b}, \bar{z}))$ *if and only if* $E(\bar{a}, \bar{b})$.

If we denote by $\mathbb{F}^{we} := (\mathbb{F}, S_1(\mathbb{F}), \{S_{2_m}(\mathbb{F})\}_{m<\omega}, \{S_{4_{m,n}}(\mathbb{F})\}_{m,n<\omega})$, then the above theorem together with [32, Proposition 4.5] implies the following:

Theorem 2.3 *Let* $\mathbb{F}$ *be a non-Abelian free group. Then* $\mathbb{F}^{we}$ *weakly eliminates imaginaries.*

Remark 2.4 *One has to be careful with stating a* $\emptyset$*-definable (i.e. definable by a first-order formula without any parameters) version of Theorem 2.2. Actually, it is easy to find a counterexample if one replaces definable by* $\emptyset$*-definable everywhere in the above theorem: let* E *be a* $\emptyset$*-definable equivalence relation with finitely many classes; then by [19, Theorem 3.1] each class is* $\emptyset$*-definable, but then the above relation can only assign to each class the single tuple consisting of trivial (imaginary) elements, i.e.* $[(1,1,1)]_{E_{4_{m,n}}}$ *or* $[(1,1)]_{E_{2_n}}$ *or* $[1]_{E_1}$ *or a trivial real element, a contradiction since we want the relation to distinguish between classes.*

Not long after the positive solution to Tarski's question (see [30, 8]), that is, whether non-Abelian free groups share the same common theory, Sela proved the following astonishing result [31].

Theorem 2.5 *The first-order theory of non-Abelian free groups,* T_{fg}*, is stable.*

We note that, by the work of Poizat [26], T_{fg} is connected, i.e. there is no definable proper subgroup of finite index (in any model of T_{fg}). In stable theories this is equivalent to saying that there is a unique generic type over any set of parameters.

We now recall some results about forking independence in the theory of the free group. For the purpose of this paper the following theorems of Pillay concerning forking independence and generic elements are enough. We denote by $\mathbb{F}_n$ the free group of rank n.

Theorem 2.6 (Corollary 2.7(ii)[23]) *Let* $n > 1$*. For any basis,* $a_1, \ldots, a_n$*, of* $\mathbb{F}_n$ *we have that* $a_1, \ldots, a_n$ *is an independent set of realizations of the (unique) generic type.*

Recall that an element of a free group $\mathbb{F}$ is called *primitive* if it is part of some basis of $\mathbb{F}$.

Theorem 2.7 (Theorem 2.1(i)[24]) *Let* $n > 1$*. Suppose a is a generic element in* $\mathbb{F}_n$*. Then a is primitive.*

2.4. Amalgamated free products

In this subsection we recall some well-known facts about amalgamated free products; we refer the reader to [10, Chapter IV] or to [11, Section 4.4] for more details and motivation. We fix 2 groups A, B a subgroup C of A and an embedding $f : C \to B$. Then the *amalgamated free product* $G := A *_C B$ is the group $\langle A, B | c = f(c), \ c \in C \rangle$. Note that G can be viewed as the free product $A * B$ quotiented by the normal subgroup generated by $\{cf(c)^{-1} \mid c \in C\}$. This construction naturally arises in the context of algebraic topology, for example in the Seifert–van Kampen theorem (see [5, Section 1.2]).

Definition 2.8 (Reduced forms) *A product of elements* $g_1 \cdot \ldots \cdot g_n$ *from* $A * B$ *for* $n \geq 0$ *is in reduced form if the following conditions hold:*

- *for each* $i \leq n$*,* $g_i \in A \cup B$ *and* g_i, g_{i+1} *belong to different factors;*

- *if $n > 1$, then no g_i belongs to C or $f(C)$;*

- *if $n = 1$, then $g_1 \neq 1$.*

Clearly, any element $g \in G$ can be written as a product of elements in reduced form, but this form is not unique.

We can obtain uniqueness once we fix systems of representatives for the right cosets of C in A and for the right cosets of $f(C)$ in B.

Definition 2.9 (Normal forms) *Let S (respectively T) be a system of right coset representatives for C in A (respectively a system of right coset representatives for $f(C)$ in B). Then a product of elements $c \cdot g_1 \cdot \ldots \cdot g_n$ from $A * B$ is in normal form if $c \in C$ and $g_1 \cdot \ldots \cdot g_n$ is in reduced form with each g_i belonging to $S \cup T$.*

We then have:

Theorem 2.10 (Normal form theorem) *Let S (respectively T) be a system of right coset representatives for C in A (respectively a system of right coset representatives for $f(C)$ in B). Let $g \in G$. Then g can be uniquely represented as a product of elements in normal form.*

Since we will do many calculations with normal forms, we give more details for a situation that will often occur. Fix S (respectively T) a system of right coset representatives for C in A (respectively $f(C)$ in B). If $g \in A$ (respectively $g \in B$), denote by $\hat{g}$ (respectively $\tilde{g}$) the element in S (respectively in T) such that $Cg = C\hat{g}$ (respectively $f(C)g = f(C)\tilde{g}$). Let $\gamma = cg_1 g_2 \ldots g_n$ be an element in normal form and let a be an element in A. We would like to calculate the normal form of $\gamma \cdot a$. We take cases with respect to whether g_n is in A or B:

- Suppose that g_n is in A. Then the normal form of $\gamma \cdot a$ is $c \cdot c_1 \widehat{g_1 c_2} \widetilde{g_2 c_3} \ldots \widehat{g_{n-1} c_n} \ \widehat{g_n a}$, where the c_is belong to C and $g_n a = c_n \widehat{g_n a}$, $g_{n-1} c_n = c_{n-1} \widetilde{g_{n-1} c_n}$, ..., $g_1 c_2 = c_1 \widehat{g_1 c_2}$.

- Suppose that g_n is in B. In this case, if a is in C, then $\gamma \cdot a$ is $c \cdot c_1 \widehat{g_1 c_2} \widetilde{g_2 c_3} \ldots \widehat{g_{n-1} c_n} \ \widetilde{g_n a}$, where the c_is belong to C and $g_n a = c_n \widetilde{g_n a}$, $g_{n-1} c_n = c_{n-1} \widehat{g_{n-1} c_n}$, ..., $g_1 c_2 = c_1 \widehat{g_1 c_2}$.

  If a is not in C, then $\gamma \cdot a$ has the following normal form: $c \cdot c_1 \widehat{g_1 c_2} \widetilde{g_2 c_3} \ldots \widehat{g_{n-1} c_n} \ \widetilde{g_n c_{n+1}} \widehat{a}$, with $a = c_{n+1} \widehat{a}$ and the obvious equations for the rest.

A product in reduced form, $g_1 \cdot \ldots \cdot g_n$, is called *cyclically reduced*, if any cyclic permutation of the g_is gives a product in reduced form. Equivalently, $g_1 \cdot \ldots \cdot g_n$ is cyclically reduced if $n = 1$ or n is even. We moreover have:

Theorem 2.11 (Conjugacy theorem for amalgamated free products) *Every element of G is conjugate to an element that can be represented as a product in a cyclically reduced form.*

Moreover, if $g := g_1 \cdot \ldots \cdot g_n, h := h_1 \cdot \ldots \cdot h_m$ are products in cyclically reduced form, which are conjugates in G. Then:

(i) *if $n = 1$ and $g \in (A \cup B) \setminus C$, then $m = 1$, h belongs to the same factor as g and they are conjugates by an element of this factor;*

(ii) *if $n = 1$ and $g \in C$, then $m = 1$ and there is a sequence of elements $g, g_1, \ldots, g_l, h$ where $g_i \in C$ and consecutive elements in the sequence are conjugates in a factor;*

(iii) *if $n > 1$, then $n = m$ and h can be obtained from g by a cyclic permutation of $g_1, \ldots, g_n$ and then conjugation by an element of C.*

2.5. Pseudo-Anosov homeomorphisms

A homeomorphism, h, of a (compact) surface Σ is called *pseudo-Anosov* if there exist a pair of transverse measured foliations, $(\mathcal{F}^u, \mu_u), (\mathcal{F}^s, \mu_s)$, and a real number $\lambda > 1$, such that h "respects" the foliations in the following sense:

$$h.(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda \cdot \mu_u) \text{ and } h.(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda^{-1} \cdot \mu_s).$$

The (isotopy) classes of pseudo-Anosov homeomorphisms play an important role in the study of the mapping class group $\mathcal{MCG}(\Sigma)$ of a (compact) surface Σ, i.e. the group of isotopy classes of orientation preserving homeomorphisms of Σ (fixing the boundary components pointwise). Let us also note that examples of pseudo-Anosov homeomorphisms were first considered by Nielsen (see [13, 14, 15]) but more sytematically studied after the work of Thurston (see [35]), where he stated the following celebrated theorem.

Theorem 2.12 (Nielsen–Thurston classification theorem) *Let Σ be a (compact) surface. Let $h \in \mathcal{MCG}\Sigma)$. Then h is either periodic, or reducible or pseudo-Anosov.*

For motivating the definition of a pseudo-Anosov, one might consider the case of an Anosov homeomorphism of the torus. We identify the torus with $\mathbb{R}^2/\mathbb{Z}^2$. The mapping class group of the torus is isomorphic to $SL_2(\mathbb{Z})$ and an Anosov homeomorphism would be a matrix $A \in SL_2(\mathbb{Z})$ with $|trace(A)| > 2$. For such a matrix, we have 2 real eigenvalues, $\lambda > 1$ and λ^{-1}, and the corresponding eigenlines in $\mathbb{R}^2$ have irrational slope. Moreover, one of the eigenlines is "stretched" by a factor of λ while the other is "contracted" by a factor of λ^{-1}. For each eigenline, the lines parallel to it form a foliation of $\mathbb{R}^2$ and the 2 foliations corresponding to the distinct eigenlines are transverse at each point. Since $\mathbb{Z}^2$ acts on the set of parallel lines, the foliations project to foliations of the torus, where each "leaf", i.e. the image of a line, is dense in T^2 and A leaves each of the foliations invariant. For a more thorough exposition of the above notions and results we refer the reader to [3, Chapter 13] or [4].

We now collect some useful properties of pseudo-Anosov homeomorphisms, which we believe to be well known, in the following theorem. We still sketch a proof that will be rather quick and hard to follow for the reader lacking geometric background.

Theorem 2.13 *Let $\Sigma_{g,1}$ be the orientable surface of genus g with connected (nonempty) boundary component. Let $\pi_1(\Sigma_{g,1}, *)$ be the fundamental group of $\Sigma_{g,1}$ with respect to the base point $*$, and let B be a maximal boundary subgroup.*

Suppose h is a pseudo-Anosov homeomorphism of $\Sigma_{g,1}$ and $[h_*]$ is the corresponding outer automorphism of $\pi_1(\Sigma_{g,1}, *)$. Then:

(i) *if $a \in \pi_1(\Sigma_{g,1}, *)$ cannot be conjugated to an element in the boundary subgroup B, then $\{[h_*]^k.[a] \mid k \in \omega\}$ is infinite, where $[a]$ denotes the conjugacy class of a;*

(ii) if $h_* \in [h_*]$ is an automorphism of $\pi_1(\Sigma_{g,1}, *)$ that fixes the boundary subgroup B, then the orbit of double cosets $B.a.B$ under powers of h_*, $\{B.h_*^k(a).B \mid k \in \omega\}$, which is infinite for any $a \notin B$.

Moreover, both (i) and (ii) hold for any infinite subsequence of powers of $[h_*]$.

Proof [Sketch of proof]

For both parts of the theorem we will use the following fact: Any element, $a \in \pi_1(\Sigma_{g,1})$, that cannot be conjugated to an element in B has uniform exponential growth under powers of h_*, i.e. there exists a "stretching factor" $\lambda_{h_*} > 1$ such that

$$\left|\widehat{h_*^k(a)}\right|_{\mathbb{F}_{2g}} \sim C_a \cdot \lambda_{h_*}^k,$$

where C_a is a constant depending only on the element a (and the choice of the generating set for $\mathbb{F}_{2g}$), and $\hat{a}$ denotes the cyclically reduced element (up to cyclic permutation) in the conjugacy class of a.

Part (i) follows immediately from the above fact.

For (ii), we consider the action of $\pi_1(\Sigma_{g,1})$ on a based real tree (T, x) obtained by the Bestvina–Paulin method (see [1, 17]) from the sequence of automorphisms $(h_*^k)_{k \in \omega} : \pi_1(\Sigma_{g,1}) \to \mathbb{F}_{2g}$ (or any infinite subsequence). Using the above fact one can easily verify the following properties of the limiting action. First, if an element $a \in \pi_1(\Sigma_{g,1})$ cannot be conjugated to an element in B, then a acts hyperbolically on T; moreover the translation length of $h_*^k(a)$, $tr_T(h_*^k(a))$, goes to infinity, as $k \to \infty$. Second, any nontrivial element of B fixes exactly x.

Now, suppose for the sake of contradiction, that for some $a \notin B$ we have that $B.h_*^k(a).B = B.a.B$ for arbitrarily large k. Then we clearly have that $tr_T(h_*^k(a)) = tr_T(b^{n_k} a b^{m_k}) = tr_T(ab^{m_k - n_k})$ and $|m_k - n_k| \to \infty$, as $k \to \infty$. We take the following cases:

Case 1 Suppose that a can be conjugated to an element in B; then a fixes a point in T that is different from x. Thus, $tr_T(a \cdot b^{m_k - n_k}) > 0$, but $tr_T(h_*^k(a)) = tr(a) = 0$, a contradiction.

Case 2 Suppose that a cannot be conjugated to an element in B. Then $tr_T(h_*^k(a)) \to \infty$ as $k \to \infty$, but $tr_T(ab^{m_k - n_k})$ is bounded by $d(x, a \cdot x)$, a contradiction.

$\square$

The same is true for nonorientable surfaces.

Theorem 2.14 *Let $\Pi_{n,1}$ be the connected sum of n projective planes with connected (nonempty) boundary component. Let $\pi_1(\Pi_{n,1}, *)$ be the fundamental group of $\Pi_{n,1}$ with respect to the base point $*$, and let B be a maximal boundary subgroup.*

Suppose that h is a pseudo-Anosov homeomorphism of $\Pi_{n,1}$ and $[h_*]$ is the corresponding outer automorphism of $\pi_1(\Pi_{n,1}, *)$. Then:

(i) if $a \in \pi_1(\Pi_{n,1}, *)$ cannot be conjugated to an element in the boundary subgroup B, then $\{[h_*]^k.[a] \mid k \in \omega\}$ is infinite, where $[a]$ denotes the conjugacy class of a;

(ii) if $h_* \in [h_*]$ is an automorphism of $\pi_1(\Pi_{n,1}, *)$ that fixes the boundary subgroup B, then the orbit of double cosets $B.a.B$ under powers of h_*, $\{B.h_*^k(a).B \mid k \in \omega\}$, which is infinite for any $a \notin B$.

Moreover, both (i) and (ii) hold for any infinite subsequence of powers of $[h_*]$.

We note that most surfaces support pseudo-Anosov homeomorphisms.

Fact 2.15 (cf. [18]) *Let Σ be either the torus with connected boundary or a (possibly nonorientable) surface with Euler characteristic at most -2; then it carries a pseudo-Anosov homeomorphism.*

3. Witnessing ampleness

In this section we prove the main result of the paper. We will show that the following sequence in $\mathbb{F}_\omega :=$ $\langle e_1, e_2, \ldots, e_k, \ldots \rangle$ witnesses n-ampleness for any $n \in \omega$ (after adding e_1, e_2 as parameters). We give the sequence recursively:

$$a_0 = e_3,$$

$$a_{i+1} = a_i[e_{2i+4}, e_{2i+5}], \text{ for } i \in \omega.$$

We fix a natural number $n \geq 1$, and we show that $a_0, \ldots, a_n$ witnesses n-ampleness by verifying the requirements of Definition 1.3.

We can now proceed with the proofs of the first 3 requirements of Definition 1.3.

Proposition 3.1 $a_0 = e_3$ *forks with* $a_n = e_3[e_4, e_5] \ldots [e_{2n+2}, e_{2n+3}]$ *over* e_1, e_2.

Proof Suppose not. Note that since a_n together with e_1, e_2 can be completed to form a basis of $\mathbb{F}_{2n+3}$, we have that a_n is generic over e_1, e_2. But then, by our contradictory hypothesis, a_n is generic over e_1, e_2, e_3. Consequently, $e_3^{-1}a_n = [e_4, e_5] \ldots [e_{2n+2}, e_{2n+3}]$ is generic over e_1, e_2, e_3 and, by Theorem 2.7, $[e_4, e_5] \ldots$ $[e_{2n+2}, e_{2n+3}]$ is a primitive element. This is a contradiction since a primitive element of $\mathbb{F}_n$ maps to a primitive element in the abelianization $\mathbb{Z}^n = \mathbb{F}_n/[\mathbb{F}_n, \mathbb{F}_n]$. □

Proposition 3.2 *Let* $1 \leq i < n$. *Then* $a_0, \ldots, a_{i-1}$ *does not fork with* a_{i+1} *over* e_1, e_2, a_i.

Proof We first note that for each i, $\langle e_1, e_2, a_i \rangle$ is a free factor of $\mathbb{F}_{2i+5}$, i.e. e_1, e_2, a_i extends to a basis of $\mathbb{F}_{2i+5}$. Thus, by Theorem 2.6, we only need to find a free factorization $\mathbb{F}_{2i+5} = \mathbb{F} * \langle e_1, e_2, a_i \rangle * \mathbb{F}'$, such that $a_0, \ldots, a_{i-1}$ is in $\mathbb{F} * \langle e_1, e_2, a_i \rangle$ and a_{i+1} is in $\langle e_1, e_2, a_i \rangle * \mathbb{F}'$. It is easy to see that the following free factorization is as follows:

$$\mathbb{F}_{2i+5} = \langle e_4, e_5, \ldots, e_{2i+2}, e_{2i+3} \rangle * \langle e_1, e_2, a_i \rangle * \langle e_{2i+4}, e_{2i+5} \rangle.$$

□

Proposition 3.3 $\mathbb{F}_3^{eq} \cap \langle e_1, e_2, e_3[e_4, e_5] \rangle^{eq} = \mathbb{F}_2^{eq}$.

Proof Let $A := \langle e_1, e_2, e_3[e_4, e_5] \rangle$ and $\alpha \in \mathbb{F}_3^{eq} \cap A^{eq}$. There then exists an equivalence relation E and a tuple $\bar{a}$ consisting of elements of $\mathbb{F}_5$ such that $\alpha = [\bar{a}]_E$. Thus, by Theorem 2.2, we have that $R_E(\bar{a}, \mathbb{F}_5^{eq}) = \{\bar{\alpha}_1, \ldots, \bar{\alpha}_k\}$. Note that each $\bar{\alpha}_i$ is algebraic over $\mathbb{F}_2\alpha$, and thus they all belong to $\mathbb{F}_3^{eq} \cap A^{eq}$. Now let β be an element of the tuple $\bar{\alpha}_i$ for some $i \leq k$. We take cases for β.

(i) Suppose $\beta \in \mathbb{F}_5$. Then $\beta \in \mathbb{F}_3 \cap A$, which is exactly $\mathbb{F}_2$.

(ii) Suppose $\beta = [b]_E$, with $E = E_1$. We may assume that b is in a cyclically reduced form with respect to the free splitting $\mathbb{F}_2 * \langle e_3, e_4, e_5 \rangle$. Since $\beta \in \mathbb{F}_3^{eq}$ (respectively $\beta \in A^{eq}$), there is $b_1 \in \mathbb{F}_3$ (respectively $b_2 \in A$) such that $[b_1]_E = [b_2]_E = [b]_E$. However, a cyclically reduced form for b_1 with respect to $\mathbb{F}_2 * \langle e_3 \rangle$ (respectively for b_2 with respect to $\mathbb{F}_2 * \langle e_3[e_4, e_5] \rangle$), will automatically be a cyclically reduced form with respect to $\mathbb{F}_2 * \langle e_3, e_4, e_5 \rangle$. Therefore, b_1 is a cyclic permutation of b_2, which implies that $b \in \mathbb{F}_3 \cap A$, as we wanted.

(iii) Suppose $\beta = [(b_1, b_2)]_E$, with $E = E_{2_m}$. If b_2 is the identity element then the result holds trivially, and thus we may assume that $b_2 \neq 1$. Since $\beta \in \mathbb{F}_3^{eq}$ (respectively $\beta \in A^{eq}$), there is $b_{21} \in \mathbb{F}_3$ (respectively $b_{22} \in A$) such that $C_{\mathbb{F}_5}(b_2) = C_{\mathbb{F}_5}(b_{21}) = C_{\mathbb{F}_5}(b_{22})$. Therefore, there are $b \in \mathbb{F}_5$ and $k, l, p \in \mathbb{Z} \setminus \{0\}$ such that $b^k = b_{21}$, $b^l = b_{22}$ and $b^p = b_2$. But, since $\mathbb{F}_3$ (respectively A) is a free factor of $\mathbb{F}_5$, if some power of an element of $\mathbb{F}_5$ belongs to $\mathbb{F}_3$ (respectively to A), then the element itself belongs to $\mathbb{F}_3$ (respectively A), and thus $b \in \mathbb{F}_3 \cap A$ and consequently $b_2 \in \mathbb{F}_2$.

Moreover, there are $b_{11} \in \mathbb{F}_3$ and $b_{12} \in A$ such that $b_1.C(b_2) = b_{11}.C(b_2) = b_{12}.C(b_2)$, and thus $b_{11} b_{12}^{-1} \in \mathbb{F}_2$ and consequently all b_1, b_{11}, b_{12} belong to $\mathbb{F}_2$.

(iv) Suppose $\beta = [(b_1, b_2, b_3)]_E$, with $E = E_{4_{m,n}}$. Then the proof that b_1, b_2, b_3 belong to $\mathbb{F}_2$ is identical to the previous case.

This shows that $\alpha \in \mathbb{F}_2^{eq}$, as we wanted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In order to verify the fourth requirement of Definition 1.3, we first need some preparatory lemmata. We formalize a construction that will often occur and point out some easy observations in the following remark:

Remark 3.4

- *We first realize $\mathbb{F}_{2g}$ as the fundamental group, $\pi_1(\Sigma_{g,1})$, of the orientable surface of genus $g > 0$ with connected boundary. Let B be a fixed boundary subgroup of $\pi_1(\Sigma_{g,1})$. Suppose A is a group and $f : B \to A$ is an injective morphism.*

  We consider the amalgamated free product $G = \mathbb{F}_{2g} *_B A$ of $\mathbb{F}_{2g}$ with A over $\{B, f\}$ and an automorphism $\alpha \in \mathrm{Aut}_B(\mathbb{F}_{2g})$ coming from a pseudo-Anosov homeomorphism. Then α extends to an automorphism fixing (pointwise) A, and we will call such an automorphism an extension of a pseudo-Anosov homeomorphism.

  More generally, any representative (in the outer class) of an outer automorphism corresponding to a homeomorphism (fixing the boundary pointwise) can be extended to an automorphism of G, restricting to conjugation on A.

- *The subgroup $A := \langle e_1, e_2, e_3, [e_4, e_5], \dots, [e_{2i+4}, e_{2i+5}] \rangle$ of $\mathbb{F}_{2i+5}$ is root closed, i.e. if, for some $a \in \mathbb{F}_{2i+5}$, $a^m \in A$, then $a \in A$. This is not hard to see either by using normal forms for free products or more elegantly by using Bass–Serre theory (for the basic notions of Bass–Serre theory we refer the reader to [33]). Consider the action on a simplicial tree corresponding to the graph of groups decomposition of the left-hand side of Figure 1.*

  We observe that the edge stabilizers for the action are root closed in $\mathbb{F}_{2i+5}$: an edge stabilizer will be a conjugate of an edge group indicated on the left side of Figure 1. We also note that an element of $\mathbb{F}_{2i+5}$

can either be elliptic or hyperbolic with respect to its action on the Bass–Serre tree. An element is elliptic if it fixes a point in the tree and hyperbolic if not; in the latter case, the element admits an invariant axis (i.e. a subtree isometric to the real line) on which it acts by translations. If an element is hyperbolic, then any of its nontrivial powers are hyperbolic with the same axis.

Now, since a^m belongs to A, it fixes the vertex stabilized by A, called x, and it is elliptic by definition. By our discussion above, a must also be elliptic. Assume, for a contradiction, that the element a fixes a vertex different from x; call it y. Then a^m fixes the segment between x and y. Thus, a^m fixes an edge adjacent to x. Since edge stabilizers are root closed, a fixes the same edge, and therefore it must fix x, a contradiction.

- *The subgroup $\langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}], [e_{2i+4}, e_{2i+5}][e_{2i+6}, e_{2i+7}] \rangle$ of $\mathbb{F}_{2i+7}$ is root closed. The arguments of the previous point, considering the graph of groups decomposition on the left-side of Figure 2, are also valid in this case.*
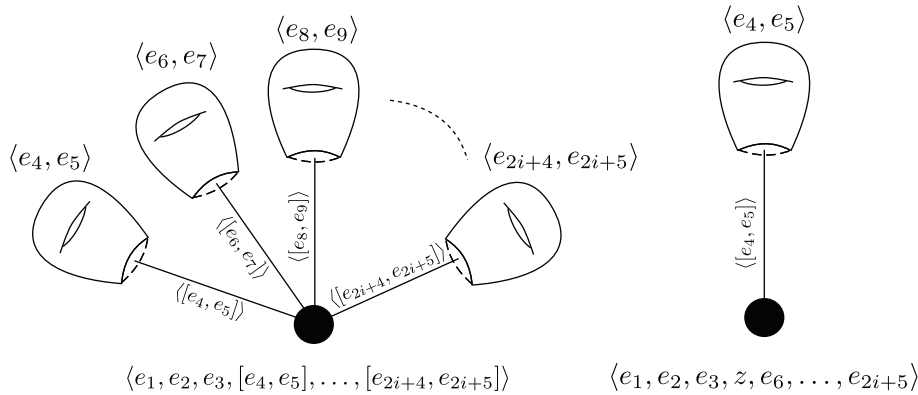

Figure 1. A series of amalgamated free products (left side). An amalgamated free product (right side).

Lemma 3.5 *Suppose $\gamma \in acl^{eq}(e_1, e_2, a_0, a_1, \ldots, a_{i+1}) \cap \mathbb{F}_{2i+5}^{we}$, for some $i < n$. Then:*

- *if γ is a real element, then $\gamma \in \langle e_1, e_2, \ a_0, a_1, \ldots, a_{i+1} \rangle$;*

- *if $\gamma = [\bar{c}]_E$ for some basic equivalence relation E, then there is $\bar{d} \in \langle e_1, e_2, a_0, a_1, \ \ldots, a_{i+1} \rangle$ with $\bar{c} \sim_E \bar{d}$.*

Proof   First we note that $\langle e_1, e_2, a_0, a_1, \ldots, a_{i+1} \rangle = \langle e_1, e_2, e_3, [e_4, e_5], \ldots, [e_{2i+4}, e_{2i+5}] \rangle$ and we denote this group by A. We take cases for γ.

(i) Let γ be an element of $\mathbb{F}_{2i+5}$. Suppose, for the sake of contradiction, that $\gamma \notin A$. We consider $\mathbb{F}_{2i+5}$ as the following finite sequence of amalgamated free products (see Figure 1):

$$(((A *_{[e_{2i+4}, e_{2i+5}]} \langle e_{2i+4}, e_{2i+5} \rangle) *_{[e_{2i+2}, e_{2i+3}]} \langle e_{2i+2}, e_{2i+3} \rangle) \ldots) *_{[e_4, e_5]} \langle e_4, e_5 \rangle.$$

We denote by A_1 the amalgamated free product $A *_{[e_{2i+4}, e_{2i+5}]} \langle e_{2i+4}, e_{2i+5} \rangle$ and by A_{l+1} the amalgamated free product $A_l *_{[e_{2i+4-2l}, e_{2i+5-2l}]} \langle e_{2i+4-2l}, e_{2i+5-2l} \rangle$ for $1 \le l \le i$.

An easy induction shows that there is $l \le i$, and so γ admits a normal form, $\delta \gamma_1 \gamma_2 \ldots \gamma_m$, with respect to the amalgamated free product A_l, for which there is γ_j for some $j \le m$ that does not belong to

A_{l-1}. Thus, without loss of generality we may assume that this is true for $l = i$ and γ_j is an element of $A_i *_{\langle [e_4,e_5] \rangle} \langle e_4, e_5 \rangle$ that does not belong to $A_i = \langle e_1, e_2, e_3, [e_4, e_5], e_6, \ldots, e_{2i+5} \rangle$.

We will obtain a contradiction by showing that γ has infinite orbit under $Aut_A(\mathbb{F}_{2i+5})$.

We realize the group $\langle e_4, e_5 \rangle$ as the fundamental group of the torus with connected boundary and we fix $\langle [e_4, e_5] \rangle$ to be the preferred boundary subgroup. Let $h_* \in Aut_A(\mathbb{F}_{2i+5})$ be an extension of a pseudo-Anosov homeomorphism. By Theorem 2.13(ii), we have that for any $a \in \langle e_4, e_5 \rangle \setminus \langle [e_4, e_5] \rangle$, the left coset of the form $a \cdot \langle [e_4, e_5] \rangle$ has an infinite orbit under powers of h_*. Since γ is in $acl^{eq}(A)$ and h_* fixes A, we have that $\{h_*^k(\gamma) \mid k \in \omega\}$ is finite. Thus, for arbitrarily large k we get $h_*^k(\gamma) = \gamma$. Now, first assume that $\gamma_m \in \langle e_4, e_5 \rangle$, and then $\delta \cdot h_*^k(\gamma_1) \ldots h_*^k(\gamma_m) \neq \delta \cdot \gamma_1 \ldots \gamma_m$, since by our previous remark $h_*^k(\gamma_m)$ would represent a different right coset from γ_m, contradicting Theorem 2.10. In the case that $\gamma_m \notin \langle e_4, e_5 \rangle$ the same argument is valid for γ_{m-1}, which is necessarily in $\langle e_4, e_5 \rangle$.

(ii) Let $\gamma = [c]_E$, for $E = E_1$. Suppose, for the sake of contradiction, that c cannot be conjugated to an element in A. As in Case (i), but using cyclically reduced forms now, we may assume that c admits a cyclically reduced form, $\gamma_1 \gamma_2 \ldots \gamma_m$, with respect to the amalgamated free product $A_i *_{[e_4,e_5]} \langle e_4, e_5 \rangle$ so that for some $j \leq m$ we have that γ_j does not belong to $A_i = \langle e_1, e_2, e_3, [e_4, e_5], e_6, \ldots, e_{2i+5} \rangle$.

We will obtain a contradiction by showing that the conjugacy class of c has infinite orbit under $Aut_A(\mathbb{F}_{2i+5})$. We consider $h_* \in Aut_A(\mathbb{F}_{2i+5})$ to be an extension of a pseudo-Anosov homeomorphism obtained exactly as in Case (i). As before, since $\{[h_*^k(c)] \mid k \in \omega\}$ is finite, c is a conjugate of $h_*^k(c)$ for arbitrarily large k. We now take cases for the length of the cyclically reduced form, $\gamma_1 \ldots \gamma_m$, for the element c. Note that we cannot have $c \in \langle [e_4, e_5] \rangle$.

- Suppose $m = 1$ (thus $\gamma_1 \in \langle e_4, e_5 \rangle \setminus \langle [e_4, e_5] \rangle$), and let I be the infinite subset of ω for which for any $k \in I$, $h_*^k(\gamma_1)$ is a conjugate of γ_1. Then, by Theorem 2.11(i), $h_*^k(\gamma_1)$ and γ_1 are conjugates in $\langle e_4, e_5 \rangle$, but, by Theorem 2.13(i), $\{[h_*^k(\gamma_1)]_E \mid k \in I\}$ is infinite, a contradiction.

- Suppose $m > 1$, and let $h_*^k(\gamma_1 \ldots \gamma_m)$ be a conjugate of $\gamma_1 \ldots \gamma_m$ for arbitrarily large k. By Theorem 2.11(iii) we have that $h_*^k(\gamma_1) \ldots h_*^k(\gamma_m)$ is obtained from $\gamma_1 \ldots \gamma_m$ by a cyclic permutation after possibly conjugating by an element of the boundary subgroup. Thus, $b(k)^{-1} \gamma_{i_k(1)} \ldots \gamma_{i_k(m)} b(k)$ is $h_*^k(\gamma_1) \ldots h_*^k(\gamma_m)$ for some cyclic permutation $i_k \in \langle (12 \ldots m) \rangle$ and some $b(k)$ in $\langle [e_4, e_5] \rangle$, for arbitrarily large k. Clearly this contradicts Theorem 2.13(ii).

(iii) Let $\gamma = [(c_1, c_2)]_{E_{2_p}}$. Recall that γ is determined by the left coset $c_1 \cdot C^p_{\mathbb{F}_{2i+5}}(c_2)$. Suppose, for the sake of contradiction, that $c_2 \notin A$. Then, as in Case (I), we have that c_2 has infinite orbit under $Aut_A(\mathbb{F}_{2i+5})$. Let $(f_k)_{k \in \omega} \in Aut_A(\mathbb{F}_{2i+5})$ be such that $f_k(c_2) \neq f_l(c_2)$ for $k \neq l$. Since $\gamma \in acl^{eq}(A)$, we have that $(f_{k_1}(c_1), f_{k_1}(c_2)) \sim_{E_{2_p}} (f_l(c_1), f_l(c_2))$ for some $k_1 \in \omega$ and arbitrarily large l. But then $C(f_{k_1}(c_2)) = C(f_l(c_2))$ for arbitrarily large l, a contradiction since for any automorphism $f \in Aut(\mathbb{F}_{2i+5})$ and any nontrivial element $c \in \mathbb{F}_{2i+5}$, if $C(c) = C(f(c))$, then either $f(c) = c$ or $f(c) = c^{-1}$. Thus, $c_2 \in A$. In the case that c_2 is the identity element we trivially have that c_1 can be chosen to be in A, and thus we may assume that $c_2 \in A \setminus \{1\}$.

Now assume, for the sake of contradiction, that $c_1 \notin A$ and, as in Case (I), we may assume that c_1 admits a normal form, $\delta_1 \gamma_{11} \ldots \gamma_{1m}$, with respect to the amalgamated free product $A_i *_{[e_4,e_5]} \langle e_4, e_5 \rangle$ so that for some $j \leq m$ we have that γ_{1j} does not belong to $A_i = \langle e_1, e_2, e_3, [e_4, e_5], e_6, \ldots, e_{2i+5} \rangle$. We take h_* to be an extension of a pseudo-Anosov homeomorphism exactly as in Case (i). We will show that γ has an infinite orbit under powers of h_*. Suppose not, and then the set of left cosets $\{ h_*^k(c_1) \cdot C_{\mathbb{F}_{2i+5}}(c_2) \mid k \in \omega \}$ is finite. Thus, for arbitrarily large k, $h_*^k(c_1) = c_1 \cdot d^{n_k}$ for some d in A (recall that c_2 must be in A and A is root closed). Consider the normal form, $\delta_1 \beta_{11} \alpha_{11} \ldots \beta_{1m} \alpha_{1m} \beta_{1(m+1)}$, for c_1 where α_{1j} belongs to A_i (note that β_{11} or $\beta_{1(m+1)}$ might be trivial). We take further cases:

- Suppose c_1 is an element in $\langle e_4, e_5 \rangle \setminus \langle [e_4, e_5] \rangle$. First note that if d is not in the boundary subgroup $\langle [e_4, e_5] \rangle$, then $c_1 d^{n_k}$ has a different normal form from $h_*^k(c_1)$, and thus we may assume that d is in the boundary subgroup. Now this easily contradicts Theorem 2.13(ii), since it implies that $\{ B.h_*^k(c_1).B \mid k \in I \}$, for some infinite subset I of ω, is finite.

- Suppose the last element in the normal form for c_1 is not in A_i (i.e. $\beta_{1(m+1)}$ is not trivial). Then d must belong to the boundary subgroup $\langle [e_4, e_5] \rangle$; otherwise, $\delta_1 \beta_{11} \alpha_{11} \ldots \beta_{1m} \alpha_{1m} \beta_{1(m+1)} d^{n_k}$ and $\delta_1 h_*^k(\beta_{11}) \alpha_{11} \ldots h_*^k(\beta_{1m}) \alpha_{1m} \; h_*^k(\beta_{1(m+1)})$ have different normal forms. Thus, for k arbitrarily large $\beta_{1(m+1)} d^{n_k}$ represent the same right coset with respect to the boundary subgroup as $h_*^k(\beta_{1(m+1)})$, and this contradicts Theorem 2.13(ii).

- Suppose the last element in the normal form for c_1 is in A_i (i.e. $\beta_{1(m+1)}$ is trivial). Then since $\delta_1 \beta_{11} \alpha_{11} \ldots \beta_{1m} \alpha_{1m} d^{n_k}$ has the same normal form as $\delta_1 h_*^k(\beta_{11}) \alpha_{11} \ldots h_*^k(\beta_{1m}) \alpha_{1m}$, we have that $\alpha_{1m} d^{n_k} = b^{p_k} \alpha_{1m}$ for some b in the boundary subgroup $\langle [e_4, e_5] \rangle$. But then again for arbitrarily large k, $\beta_{1m} b^{p_k}$ and $h_*^k(\beta_{1m})$ should represent the same right coset with respect to the boundary subgroup, a contradiction.

(iv) Let $\gamma = [(c_1, c_2, c_3)]_{E_{4_{p,q}}}$. Recall that γ is determined by the double coset $C^p_{\mathbb{F}_{2i+5}}(c_1) \cdot c_2 \cdot C^q_{\mathbb{F}_{2i+5}}(c_3)$. We can easily see, by the proof of Case (iii), that $c_1, c_3 \in A \setminus \{1\}$. Suppose, for the sake of contradiction, that $c_2 \notin A$. If we repeat the arguments of Case (iii), we get a normal form, $\delta_2 \gamma_{21} \ldots \gamma_{2m}$, for c_2 (with respect to the above-mentioned amalgamated free product) and we reach the conclusion that for arbitrarily large k, $h_*^k(c_2) = b_1^{q_k} \cdot c_2 \cdot b_2^{p_k}$ for some elements $b_1, b_2 \in A$. Consider the normal form, $\delta_2 \beta_{21} \alpha_{21} \ldots \beta_{2m} \alpha_{2m} \beta_{2(m+1)}$, for c_2 where α_{2j} belongs to A_i. The situation is completely identical with Case (iii), but we still give the arguments for completeness. We take further cases:

- Suppose c_2 is an element in $\langle e_4, e_5 \rangle \setminus \langle [e_4, e_5] \rangle$. First note that b_1, b_2 must be in the boundary subgroup $\langle [e_4, e_5] \rangle$; otherwise, $b_1^{q_k} \cdot c_2 \cdot b_2^{p_k}$ has a different normal form from $h_*^k(c_2)$. This easily contradicts Theorem 2.13(ii), since it implies that $\{ B.h_*^k(c_2).B \mid k \in I \}$, for some infinite subset I of ω, is finite.

- Suppose the last element in the normal form for c_2 is not in A_i (i.e. $\beta_{1(m+1)}$ is not trivial). Then b_2 must belong to the boundary subgroup $\langle [e_4, e_5] \rangle$; otherwise, $b_1^{p_k} c_2 b_2^{q_k}$ has a different normal form from $h_*^k(c_2)$. Now, as before, we have that $\beta_{1(m+1)} b_2^{q_k}$ represent the same right coset with respect to the boundary subgroup as $h_*^k(\beta_{1(m+1)})$, a contradiction.

– Suppose the last element in the normal form for c_2 is in A_i (i.e. $\beta_{1(m+1)}$ is trivial). Since $b_1^{p_k}\delta_2\beta_{21}\alpha_{21}\ldots\beta_{2m}\alpha_{2m}b_2^{q_k}$ has the same normal form as $\delta_2 h_*^k(\beta_{21})\alpha_{21}\ldots h_*^k(\beta_{2m})\alpha_{2m}$, we have that $\alpha_{2m}b_2^{q_k} = b^{l_k}\alpha_{2m}$ for some element b in the boundary subgroup $\langle[e_4,e_5]\rangle$. Now if $m > 1$, the proof is identical to the analogous part in the proof of Case (iii). If $m = 1$, then c_2 has the form $\delta_2\beta\alpha$. Thus, $b_1^{p_k}\delta_2\beta b^{l_k}\alpha$ has the same normal form as $\delta_2 h_*^k(\beta)\alpha$, so $b_1^{p_k}$ must be in the boundary subgroup. But then again, since $b_1^{p_k}\delta_2$ cannot change the coset, we have that βb^{l_k}, $h_*^k(\beta)$ represent the same right coset with respect to the boundary subgroup, contradicting Theorem 2.13(ii). $\qquad\square$

Lemma 3.6 *Suppose $\gamma \in acl^{eq}(e_1, e_2, a_0, a_1, \ldots, a_i, a_{i+2}) \cap \mathbb{F}_{2i+7}^{we}$, for some $i < n$. Then:*

- *if γ is a real element, then $\gamma \in \langle e_1, e_2,\ a_0, a_1, \ldots, a_{i+2}\rangle$;*

- *if $\gamma = [\bar{c}]_E$ for some basic equivalence relation E, then there is $\bar{d} \in \langle e_1, e_2, a_0, a_1,\ \ldots, a_{i+2}\rangle$ with $\bar{c} \sim_E \bar{d}$.*

Proof In this case we have that

$$\langle e_1, e_2, a_0, a_1, \ldots, a_i, a_{i+2}\rangle = \langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}], [e_{2i+4}, e_{2i+5}][e_{2i+6}, e_{2i+7}]\rangle.$$

The proof is identical to the proof of Lemma 3.5 and is left to the reader. A hint is to use homeomorphisms of $\Sigma_{1,1}$ as well as homeomorphisms of $\Sigma_{2,1}$ (see Figure 2). Note that the Euler characteristic of $\Sigma_{2,1}$ is -3, and thus it carries a pseudo-Anosov. $\qquad\square$
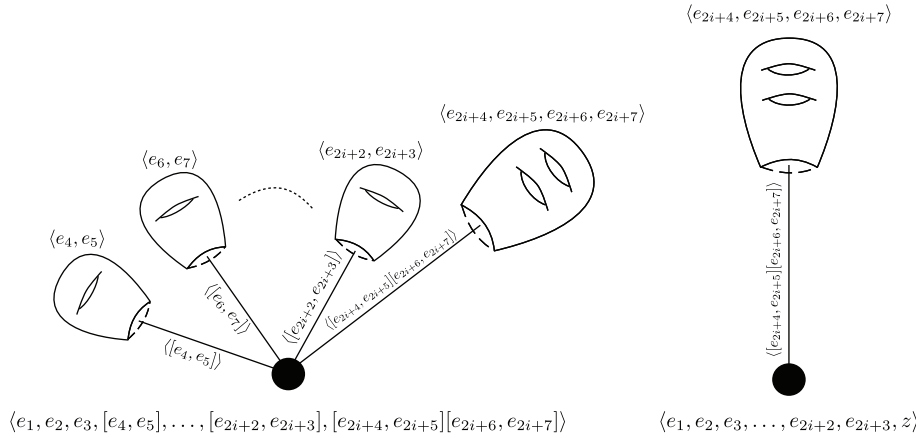

Figure 2. A series of amalgamated free products (left side). An amalgamated free product (right side).

We are now ready to verify the fourth requirement of Definition 1.3.

Proposition 3.7 *Let $0 \le i < n$. Then $acl^{eq}(e_1, e_2, a_0, \ldots, a_i, a_{i+1}) \cap acl^{eq}(e_1, e_2, a_0,\ \ldots, a_i, a_{i+2}) = acl^{eq}(e_1, e_2, a_0, \ldots, a_i)$.*

Proof We denote by A (respectively B) the group $\langle e_1, e_2, e_3, [e_4, e_5], \ldots, [e_{2i+4}, e_{2i+5}]\rangle$ (respectively $\langle e_1, e_2, e_3, [e_4, e_5], \ldots, [e_{2i+4}, e_{2i+5}][e_{2i+6}, e_{2i+7}]\rangle$). Let $\gamma \in acl^{eq}(A) \cap acl^{eq}(B)$. Then there exists an equivalence relation E and a tuple $\bar{c}$ consisting of elements of $\mathbb{F}_{2i+7}$ such that $\gamma = [\bar{c}]_E$. Thus, by Theorem 2.2, we have that $R_E(\bar{c}, \mathbb{F}_{2i+7}^{eq}) = \{\bar{\gamma}_1, \ldots, \bar{\gamma}_k\}$. Note that each $\bar{\gamma}_i$ is algebraic over $\mathbb{F}_2\gamma$, and thus they all belong to $acl^{eq}(A) \cap acl^{eq}(B)$. Now let β be an element of the tuple $\bar{\gamma}_i$ for some $i \le k$. We take cases for β.

(i) Suppose $\beta \in \mathbb{F}_{2i+7}$. Then by Lemmata 3.5 and 3.6, we have that $\beta \in A \cap B$. Consider the free splitting of $\mathbb{F}_{2i+7}$ as $\langle e_1, e_2, \ldots, e_{2i+3}\rangle * \langle e_{2i+4}, \ldots, e_{2i+7}\rangle$. Let $c_1 b_1 \ldots c_n b_n c_{n+1}$ be the normal form for β with respect to this splitting of $\mathbb{F}_{2i+7}$ where $c_i \in \langle e_1, e_2, \ldots, e_{2i+3}\rangle$ and $b_i \in \langle e_{2i+4}, \ldots, e_{2i+7}\rangle$ for $i \leq n$. Since $\beta \in A \cap B$, we must have that $b_i \in \langle [e_{2i+4}, e_{2i+5}]\rangle \cap \langle [e_{2i+4}, e_{2i+5}][e_{2i+6}, e_{2i+7}]\rangle$, but then the b_is are trivial and $\beta \in \langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}]\rangle$ as we wanted.

(ii) Suppose $\beta = [c]_E$ for $E = E_1$. Then, by Lemmata 3.5 and 3.6, c can be conjugated to an element in A and to an element in B. Consider the free splitting of $\mathbb{F}_{2i+7}$ as $\langle e_1, e_2, \ldots, e_{2i+3}\rangle * \langle e_{2i+4}, \ldots, e_{2i+7}\rangle$. Note that A (respectively B) inherits the following free splitting $\langle e_1, e_2, e_3, [e_4, e_5] \ldots, [e_{2i+2}, e_{2i+3}]\rangle * \langle [e_{2i+4}, e_{2i+5}]\rangle$ (respectively $\langle e_1, e_2, e_3, [e_4, e_5] \ldots, [e_{2i+2}, e_{2i+3}]\rangle * \langle [e_{2i+4}, e_{2i+5}][e_{2i+6}, e_{2i+7}]\rangle$) from the above splitting of $\mathbb{F}_{2i+7}$. However, any cyclically reduced form with respect to the free splitting of A (respectively B) is a cyclically reduced form with respect to the free splitting of $\mathbb{F}_{2i+7}$. Thus, if $c_A := c_1 b_1 \ldots c_n b_n$ (respectively $c_B := d_1 f_1 \ldots d_m f_m$) is the conjugate of c in A (respectively B), c_A is a cyclic permutation of c_B, and, as in Case (i), we have that c_A belongs to $\langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}]\rangle$; thus, c can be conjugated to $\langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}]\rangle$.

(iii) Suppose $\beta = [(c_1, c_2)]_E$ for $E = E_{2_m}$. Then by Lemmata 3.5 and 3.6, we have that there exist $c_{21} \in A$ and $c_{22} \in B$ such that $C(c_2) = C(c_{21}) = C(c_{22})$. Thus, there are c and $k_1, k_2, k_3 \in \mathbb{Z}$ such that $c^{k_1} = c_2$, $c^{k_2} = c_{21}$, and $c^{k_3} = c_{22}$. But since A, B are closed under taking roots, we have that $c \in A \cap B$. Thus, as in Case (i), we have that $c \in \langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}]\rangle$, and so is c_2.

Again by Lemmata 3.5 and 3.6, we have that there exist $c_{11} \in A$ and $c_{12} \in B$ such that $c_1 \cdot C(c_2)^m = c_{11} \cdot C(c_2)^m = c_{12} \cdot C(c_2)^m$. Therefore, $c_{11}^{-1} c_1 \in C(c_2)^m$ (respectively $c_{12}^{-1} c_1 \in C(c_2)^m$); thus $c_1 \in A \cap B$, and, as in Case (I), we have that $c_1 \in \langle e_1, e_2, e_3, \ldots, [e_{2i+2}, e_{2i+3}]\rangle$, as we wanted.

(iv) Suppose $\beta = [(c_1, c_2, c_3)]_E$ for $E = E_{4_{m,n}}$. The proof is identical to the previous case.

This shows that γ is in $acl^{eq}(e_1, e_2, a_0, \ldots, a_i)$ and the proof is concluded. □

Putting everything together, Propositions 3.1, 3.2, 3.3, and 3.7 show that the sequence $(a_i)_{i<\omega}$ witnesses n-ampleness in the theory of non-Abelian free groups for any $n < \omega$.

Theorem 3.8 T_{fg} *is n-ample for any $n < \omega$.*

Our method produces a family of witnessing examples to ampleness. One can easily see that for each $k \geq 1$, a sequence of the form

$$a_0 = e_3,$$

$$a_{i+1} = a_i [e_{2ki+3+1}, e_{2ki+3+2}][e_{2ki+3+3}, e_{2ki+3+4}] \ldots [e_{2ki+3+(2k-1)}, e_{2ki+3+2k}]$$

witnesses n-ampleness for any $n < \omega$. The same is true for the following "nonorientable" witnessing family of sequences (for $k \geq 3$):

$$a_0 = e_3,$$

$$a_{i+1} = a_i e_{ki+3+1}^2 e_{ki+3+2}^2 \ldots e_{ki+3+k-1}^2 e_{ki+3+k}^2.$$

Remark 3.9 *Let us also remark that there is no harm in starting the recursive definition of our witnessing sequence with $a_0 = e_1$. The point of not doing so is that we prefer to use Theorem 2.2 instead of Theorem 2.3.*

A. Appendix

We show that the "basic" equivalence relation induced by conjugation (see Definition 2.1) cannot be (geometrically) eliminated (i.e. there exists an equivalence class that is not interalgebraic with any finite "real" tuple) in the theory of the free group, strengthening Theorem 2.1 in [32]. We also show that the "basic" equivalence relations $E_{2_m}, E_{3_m}, E_{4_{m,n}}$ in Definition 2.1 cannot be eliminated giving alternative proofs of Theorems 2.2 and 2.3 of the same preprint mentioned above. In comparison with Sela's proofs, which use the existence and properties of "Diophantine envelopes" for definable sets, we use the (also highly nontrivial) result (see [30, 8]) that the following chain of groups is elementary:

$$\mathbb{F}_2 \prec \mathbb{F}_3 \prec \ldots \prec \mathbb{F}_n \prec \ldots$$

Theorem A.1 *Fix $n \geq 2$. Then for any finite tuple $\bar{a} \in \mathbb{F}_{n+1}$, we have that $[e_{n+1}]_{E_1}$ is not interalgebraic with $\bar{a}$ over $\mathbb{F}_n$.*

Proof Suppose for the sake of contradiction that there is $\bar{a} \in \mathbb{F}_{n+1}$ such that $[e_{n+1}]_{E_1}$ is interalgebraic with $\bar{a}$ over $\mathbb{F}_n$. It is not hard to see that $\bar{a} \in \mathbb{F}_{n+1} \setminus \mathbb{F}_n$ (otherwise we can fix $\bar{a}$ and send e_{n+1} to e_{n+i} for $1 < i < \omega$).
Claim: Let $b \in \mathbb{F}_n$ and $f_b \in Aut_{\mathbb{F}_n}(\mathbb{F}_{n+1})$ be defined by $f_b(e_{n+1}) = e_{n+1}^b$. Then $\bar{a}$ has infinite orbit under $\langle f_b^l | l < \omega \rangle$.
Proof of Claim: Let $c = e_{n+1}^{i_1} w_1(e_1, \ldots, e_n) e_{n+1}^{i_2} \ldots e_{n+1}^{i_k} w_k(e_1, \ldots, e_n) e_{n+1}^{i_{k+1}}$ be the normal form of an element in the tuple $\bar{a}$, which is moreover in $\mathbb{F}_{n+1} \setminus \mathbb{F}_n$ with respect to $\mathbb{F}_n * \langle e_{n+1} \rangle$. Then $f_b^l(c) = b^l e_{n+1}^{i_1} b^{-l} w_1(e_1, \ldots, e_n) b^l e_{n+1}^{i_2} b^{-l} \ldots b^l e_{n+1}^{i_k} b^{-l} w_k(e_1, \ldots, e_n) \ b^l e_{n+1}^{i_{k+1}} b^{-l}$ and the claim follows easily.

Now, since f_b^l fixes the conjugacy class of e_{n+1}, we have $\bar{a} \notin acl^{eq}(\mathbb{F}_n, [e_{n+1}]_{E_1})$, a contradiction. □

We continue by proving that no basic imaginary can be eliminated.

Theorem A.2 *Fix $n \geq 2$. Let E be a basic equivalence relation (see Definition 2.1). Then there exists an equivalence class $[\bar{b}]_E$ such that for any $\bar{a} \in \mathbb{F}_\omega$, we have that $[\bar{b}]_E$ and $\bar{a}$ are not interdefinable over $\mathbb{F}_n$.*

Proof We take cases according to the basic equivalence relation E.

(i) Let $E = E_1$. Then the result follows from Theorem A.1.

(ii) Let $E = E_{2_m}$. Then we consider the class $[(e_{n+1}, e_{n+2})]_E$, which is determined by the left coset $e_{n+1} \cdot C_{\mathbb{F}_{n+2}}^m(e_{n+2})$. Suppose, for the sake of contradiction, that there is $\bar{a} \in \mathbb{F}_\omega$ such that $[(e_{n+1}, e_{n+2})]_E$ is interdefinable with $\bar{a}$ over $\mathbb{F}_n$. Then, as in the proof of Theorem A.1, we must have that $\bar{a} \in \mathbb{F}_{n+2} \setminus \mathbb{F}_{n+1}$. But then the automorphism of $\mathbb{F}_{n+2}$ fixing $\mathbb{F}_{n+1}$ and sending $e_{n+2} \mapsto e_{n+2}^{-1}$ fixes $[(e_{n+1}, e_{n+2})]_E$ and moves $\bar{a}$, a contradiction.

(iii) Let $E = E_{4_{m,n}}$. Then we consider the class $[(e_{n+1}, e_{n+2}, e_{n+1})]_E$ and the result follows as above.

□

Acknowledgments

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