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THE FREE PSEUDOSPACE IS N-AMPLE, BUT NOT $(N + 1)$-AMPLE

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Abstract. We give a uniform construction of free pseudospaces of dimension n extending work in [1]. This yields examples of ω-stable theories which are n-ample, but not $n + 1$-ample. The prime models of these theories are buildings associated to certain right-angled Coxeter groups.

§1. Introduction. In the investigation of geometries on strongly minimal sets the notion of ampleness plays an important role. Algebraically closed fields are n-ample for all n and it is not known whether there are strongly minimal sets which are n-ample for all n and do not interpret an infinite field. Obviously, one way of proving that no infinite field is interpretable in a theory is by showing that the theory is *not* n-ample for some n.

In [1], Baudisch and Pillay constructed a free pseudospace of dimension 2. Its theory is ω-stable (of infinite rank) and 2-ample. F. Wagner posed the question whether this example was 3-ample or not.

In Section 2 we give a uniform construction of a free pseudospace of dimension n and show that it is n-ample, but not $n + 1$-ample. It turns out that the theory of the free pseudospace of dimension n is the first order theory of a Tits-building associated to a certain Coxeter diagram and we will investigate this connection in Section 4.

In the final section we determine the orthogonality classes of regular types.

The construction given here is quite similar to the one given by Evans in [2] for a stable theory which is n-ample for all n, but does not interpret an infinite group. In contrast to the examples constructed by Evans, our theory is trivial and no infinite group is definable.

§2. Construction and results. Fix a natural number $n \geq 1$. Let L_n be the language for $n + 1$-colored graphs containing predicates $V_i, i = 0, \dots n$ and an edge relation E. If $x \in V_i$ we also say that x is of level i.

By an L_n-graph we mean an $n + 1$-colored graph with vertices of types V_i, $i = 0, \dots n$ and an edge relation $E \subseteq \bigcup_{i=1,\dots n} V_{i-1} \times V_i$. We say that a path in this graph is of type E_i if all its vertices are in $V_{i-1} \cup V_i$ and of type $E_i \cup \dots \cup E_{i+j}$ if all its vertices are in $V_{i-1} \cup \dots \cup V_{i+j}$.

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The free pseudospaces will be modeled along the lines of a projective space as a simplicial complex, i.e., we will think of vertices of type V_i as i-dimensional spaces in a free pseudospace. Therefore we extend the notion of incidence as follows:

DEFINITION 2.1.

1. We say that vertices x_i, x_j of type V_i and V_j, respectively, are *incident* (for $i \leq j$) if there are vertices x_ℓ of type $V_\ell, \ell = i+1 \dots j$, such that $E(x_{\ell-1}, x_\ell)$ holds. In this case the sequence $(x_i, \dots x_j)$ is called a *dense flag*. A *flag* is a sequence of vertices $(x_1, \dots x_k), k \geq 0$ in which any two vertices are incident (and hence no two vertices of a flag have the same level). (The levels of a flag may be increasing or decreasing.) In particular, a vertex x is incident with itself.
2. The *residue* $R(x)$ of a vertex x is the set of vertices incident with x. We write $R_<(x)$ (and $R_>(x)$, respectively) for the elements in $R(x)$ of level less (greater, respectively) than the level of x. Similarly for $R_\leq(x)$ and $R_\geq(x)$
3. We say that two vertices x and y *intersect in the vertex* z, and write $z = x \wedge y$, if $R_\leq(x) \cap R_\leq(y) = R_\leq(z)$. If $R_\leq(x) \cap R_\leq(y) = \emptyset$, we say that x and y intersect in the empty set.
4. Similarly, we say that two vertices x and y *generate the vertex* z, and write $z = x \vee y$, if $R_\geq(x) \cap R_\geq(y) = R_\geq(z)$. If $R_\geq(x) \cap R_\geq(y) = \emptyset$, we say that x and y generate the empty set.
5. A *simple cycle* is a cycle without repetitions.

We now give an inductive definition of a free pseudospace of dimension n:

DEFINITION 2.2. A free pseudospace of dimension 0 is an infinite set of vertices. Assume that a free pseudospace of dimension $n-1$ has been defined. Then a free pseudospace of dimension n is an L_n-graph such that the following holds:

$(\Sigma 1)_n$ (a) The set of vertices of type $V_0 \cup \dots \cup V_{n-1}$ is a free pseudospace of dimension $n-1$.
(b) The set of vertices of type $V_1 \cup \dots \cup V_n$ is a free pseudospace of dimension $(n-1)$.

$(\Sigma 2)_n$ (a) For any vertex x of type V_0, $R_>(x)$ is a free pseudospace of dimension $(n-1)$.
(b) For any vertex x of type V_n, $R_<(x)$ is a free pseudospace of dimension $(n-1)$.

$(\Sigma 3)_n$ (a) Any two vertices x and y intersect in a vertex or the empty set.
(b) Any two vertices x and y generate a vertex or the empty set.

$(\Sigma 4)_n$ (a) If a is a vertex of type V_0 and $b, b' \in R_>(a)$ with $b' \notin R(b)$ are connected by a path γ of length k such that for some dense flags $f = (a, \dots b)$ and $f' = (b', \dots, a)$ the concatenation of these paths $f \circ \gamma \circ f'$ is a simple cycle, there is a path γ' of length at most k from b to b' in $R_>(a)$ containing some interior vertex of γ such that $f \circ \gamma' \circ f'$ is a simple cycle.
(b) If a is a vertex of type V_n and $b, b' \in R_<(a)$ are connected by a path γ of length k such that for some dense flags $f = (a, \dots b)$ and $f' = (b', \dots, a)$ the concatenation $f \circ \gamma \circ f'$ of these paths is

a simple cycle, there is a path γ' of length at most k from b to b' in $R_<(a)$ containing some interior vertex of γ such that $f \circ \gamma' \circ f'$ is a simple cycle.

REMARK 2.3. Note that by $(\Sigma 4)$ any path $\gamma = (b = x_0, \ldots, x_m = b')$ with $b, b' \in R_<(a)$ and $m \geq 2$ contains an interior vertex which lies in $R_<(a)$ unless $b' \in R(b)$ (and dually for $R_>(a)$).

Note that a free pseudospace of dimension 1 is a free pseudoplane, i.e., an L_1-graph which by $(\Sigma 4)_1$ does not contain any cycles and such that any vertex has infinitely many neighbors.

Let T_n denote the L_n-theory expressing these axioms.

Note that the inductive nature of the definition immediately has the following consequences:

1. The induced subgraph on $V_j \cup \ldots \cup V_{j+m}$ is a free pseudospace of dimension m.
2. The notion of a free pseudospace of dimension n is *self-dual*: if we put $W_i = V_{n-i}, i = 0, \ldots n$, then $W_0, \ldots W_n$ with the same set of edges is again a free pseudospace of dimension n.
3. If $a \in V_i$, then $R_<(a)$ is a free pseudospace of dimension $i - 1$ (and dually for $R_>(a)$.)

Our first goal is to show that T_n is consistent[1] and complete.

DEFINITION 2.4. Let A be a finite L_n-graph. The following extensions are called minimal strong extensions of A:

1. Add a vertex of any type to A which is connected to at most one vertex of A of an appropriate type.
2. If $(x_0, x_1, \ldots, x_k, x_{k+1})$ is a dense flag in A, add vertices $y_1, \ldots, y_k$ such that $(x_0, y_1, \ldots, y_k, x_{k+1})$ is again a dense flag.

We say that B is a strong extension of A, written $A \leq B$, if B arises from A by a sequence of finitely many minimal strong extensions.

DEFINITION 2.5. Let $\mathcal{K}_n$ be the class of finite L_n-graphs A such that the following holds

1. The empty graph is strong in A.
2. If $a, a' \in A$ with $R_\leq(a) \cap R_\leq(a') \neq \emptyset$, then there is a unique vertex $z \in A$ with $R_\leq(a) \cap R_\leq(a') = R_\leq(z)$.
3. If $a, a' \in A$ with $R_\geq(a) \cap R_\geq(a') \neq \emptyset$, then there is a unique vertex $z \in A$ with $R_\geq(a) \cup R_\geq(a') = R_\geq(z)$.
4. If $a \in V_i$ and $b, b' \in R(a)$ are connected by a path γ of length k contained in $V_{i-m} \cup \ldots \cup V_i$ such that for some dense flags $f = (a, \ldots b)$ and $f' = (b', \ldots, a)$ the concatenation of these paths $f \circ \gamma \circ f'$ is a simple cycle, there is a path γ' from b to b' of length at most k containing some interior vertex of γ and contained in $R(a) \cap (V_{i-m} \cup \ldots \cup V_{i-1})$ such that $f \circ \gamma \circ f'$ is a simple cycle.
5. If $a \in V_i$ and $b, b' \in R(a)$ are connected by a path γ of length k contained in $V_i \cup \ldots \cup V_{i+m}$ such that for some dense flags $f = (a, \ldots b)$ and $f' = (b', \ldots, a)$

[1] Readers familiar with buildings will notice that any building of type $A_{\infty, n+1}$ with infinite valencies is a model of T_n.

the concatenation of these paths is a simple cycle, there is a path γ' from b to b' of length at most k containing some interior vertex of γ and contained in $R(a) \cap (V_{i+1} \cup \ldots \cup V_{i+m})$ such that $f \circ \gamma' \circ f'$ is a simple cycle.

6. If $a \in V_i, c \in V_{i+j}$ and $b, b' \in R(a) \cap R(c)$ are connected by a path $\gamma \subset V_i \cup \ldots V_{i+j}$ of length k, then there is a path $\gamma' \subset R_>(a) \cap R_<(c) \subseteq (V_{i+1} \cup \ldots V_{i+j-1})$ from b to b' of length at most k containing some interior vertex of γ .

Note that for the whole structure the property corresponding to 6. follows from T_n.

Remark 2.6. Note that if $A \in \mathcal{K}_n$, then $A \cap (V_i \cup \ldots \cup V_{i+j}) \in \mathcal{K}_j$. Conversely, any $A \in \mathcal{K}_j$ can be considered as a graph in $\mathcal{K}_n$ for any $n \geq j$.

We next show that $(\mathcal{K}, \leq)$ has the amalgamation property for strong extensions. This will be enough to obtain a strong limit which is well-defined up to isomorphism (see [8]).

For any finite L_n-graphs $A \subseteq B, C$ we denote by $B \otimes_A C$ the *free amalgam* of B and C over A, i.e., the graph on $B \cup C$ containing no edges between elements of $B \setminus A$ and $C \setminus A$.

Lemma 2.7. *If $A \leq B, C$ are in $\mathcal{K}_n$, then $D := B \otimes_A C \in \mathcal{K}_n$ and $B, C \leq D$.*

Proof. Clearly, $B, C \leq D$. To see that $D \in \mathcal{K}_n$, note that if $B \in \mathcal{K}_n$ and B' is a minimal strong extension of B, then also $B' \in \mathcal{K}_n$. This is clear for strong extensions of type 1. For strong extensions of type 2 suppose that we have added a dense flag $(y_1, \ldots, y_m)$ to B connecting $y_o, y_{m+1} \in B$. Conditions 1, 2, and 3 are clear. To see that Condition 4 holds, let $a \in V_i, b, b' \in R(a)$, and suppose that we have a path $\gamma = (b, \ldots, b') \subset B'$ of length k in $V_{i-m} \cup \ldots \cup V_i$ containing a new vertex y_i as in the assumptions of Condition 4. We may assume $\gamma \not\subset B$. Since the new vertices have exactly two neighbors, note that a cannot be one of the new vertices. Furthermore, either the entire flag is contained in γ and can be replaced by a flag of the same type in B or an initial or end segment of γ is contained in the flag. By symmetry assume that an end segment of γ is contained in the new flag, so $b' = y_i$ for some $i \in \{1, \ldots, m\}$. In this case the entire flag $(y_1, \ldots, y_m)$ is in $R_<(a)$, and so are y_0, y_{m+1}. By assumption we may replace the flag $(y_1, \ldots, y_m)$ by a flag in $B \cap R(a)$ connecting y_0 and y_{m+1}. By Condition 4 we find a path from b to y_0 inside $R(a) \cap B$. This can be extended to the required path from b to $b' = y_i$ inside $R(a)$. Conditions 5 and 6 are similar. ⊣

This shows that the class $(\mathcal{K}_n, \leq)$ has a Fraïssé limit M_n.

Proposition 2.8. *The Hrushovski limit M_n is a model of T_n.*

Proof. This is clear for the case $n = 1$. By construction, M_n satisfies $(\Sigma 3)_n$ and $(\Sigma 4)_n$. By Remark 2.6 and induction, $V_i \cup \ldots \cup V_{i+j}$ is a model of T_j for any $i + j < n$. In particular, M_n satisfies $(\Sigma 1)_n$. It is left to show that M_n satisfies $(\Sigma 2)_n$. So let $a \in V_n$. We have to show that $R(a)$ is a pseudospace of dimension $n-1$. Clearly, $(\Sigma 3)_{n-1}$ continues to hold. By induction applied to $V_1 \cup \ldots \cup V_n$, $R(a) \cap (V_1 \cup \ldots \cup V_{n-1})$ is a pseudospace of dimension $n-2$, so $(\Sigma 1)_{n-1}(b)$ holds. Similarly $(\Sigma 2)_{n-1}(b)$ and $(\Sigma 4)_{n-1}(b)$ hold. Note that $(\Sigma 4)_{n-1}(a)$ holds for $R(a)$ by condition 6. It is left to show that $(\Sigma 1)_{n-1}(a)$ and $(\Sigma 2)_{n-1}(a)$ hold for $R(a)$, that

is we have to show that $R(a) \cap (V_0 \cup \ldots V_{n-2})$ and $R(a) \cap R(c)$ for $c \in V_0$ are pseudospaces of dimension $n-2$.
$(\Sigma 1)_{n-1}(a)$: To see that $R(a) \cap (V_0 \cup \ldots V_{n-2})$ is a pseudospace of dimension $n-2$ note that by induction $(\Sigma 1)_{n-2}(b)$, $(\Sigma 2)_{n-2}(b)$, $(\Sigma 3)_{n-2}(a)$ and (b) and $(\Sigma 4)_{n-2}(a)$ and (b) hold and it is left to show that $R(a) \cap (V_0 \cup \ldots \cup V_{n-2})$ satisfies $(\Sigma 1)_{n-2}(a)$ and $(\Sigma 2)_{n-2}(a)$. That is we have to show that $R(a) \cap (V_0 \cup \ldots V_{n-3})$ and $R(a) \cap (V_1 \cup \ldots V_{n-2}) \cap R(c)$ for $c \in V_0$ are pseudospaces of dimension $n-3$. In this way we reduce to show that $R(a) \cap (V_0 \cup V_1)$ and $R(a) \cap (V_1 \cup V_2) \cap R(c)$ for $c \in V_0$ are pseudospaces of dimension 1, and this is obvious.
$(\Sigma 2)_{n-1}(a)$: To see that $R(a) \cap R(c)$ for $c \in V_0$ is a pseudospace of dimension $n-2$ note that $(\Sigma 2)_{n-2}(a)$ and (b), $(\Sigma 3)_{n-2}(a)$ and (b), and $(\Sigma 4)_{n-2}(a)$ and (b) hold. It is therefore left to show that $R(a) \cap (V_1 \cup \ldots V_{n-2}) \cap R(c)$ and $R(a) \cap (V_2 \cup \ldots V_{n-1}) \cap R(c)$ are pseudospaces of dimension $n-3$. The first part was shown above in the proof for $(\Sigma 1)_{n-1}(a)$ for $R(a)$, the second follows by reduction to $R(a) \cap (V_2 \cup V_3) \cap R(c)$, which is a pseudospace of dimension 1. ⊣

Note that $(\Sigma 4)$ implies the following:

Lemma 2.9. *Let M be a model of T_n, $a, c \in M$ with $a \notin R(c)$. Let $\gamma_1 = (x_0 = a, \ldots x_s), \gamma_2 = (y_0 = a, \ldots y_t)$ be paths with $x_s, y_t \in R(c)$ and let i, j be minimal with $x_i, y_j \in R(c)$. Suppose that γ_1, γ_2 do not contain any element of higher level than the level of c. Then (x_i, y_j) is a flag.*

Proof. Since $x_i, y_j \in R_<(c)$ the claim follows directly from $(\Sigma 4)$ as otherwise the path $(x_i, x_{i-1}, \ldots, x_0 = y_0, y_1, \ldots, y_j)$ would have to contain an element of $R(c)$ in its interior, which it doesn't. ⊣

Corollary 2.10. *Let M be a model of T_n, $a, c \in M$ with $a \notin R(c)$. There is a flag $C \subset R_<(c)$ such that for any $b \in R_<(c)$ and any path from a to b not containing an element of higher level than the level of c enters $R_<(c)$ via an element of C.*

From now on we work inside models of T_n unless specified otherwise.

Let us say that a path $\gamma = (a = x_0, \ldots, x_m = b)$ *changes direction in x_i* if $x_i \in V_j$ and either $x_{i-1}, x_{i+1} \in V_{j-1}$ or $x_{i-1}, x_{i+1} \in V_{j-1}$ for some j. Clearly, a path which never changes direction is a dense flag.

Definition 2.11. Let $\gamma = (y_0, \ldots, y_1, \ldots, y_{k+1})$ be a path changing direction exactly in $y_1, \ldots y_k$. We say that γ is reduced if for all $i = 0, \ldots k-1$ we have

$$y_i \vee y_{i+2} = y_{i+1} \text{ or } y_i \wedge y_{i+2} = y_{i+1}.$$

Clearly, any part of a reduced path is again reduced.

Lemma 2.12. *If $\gamma = (a, \ldots, b) \subseteq V_j \cup \ldots \cup V_{j+m}$ has length s, there is a reduced path from a to b inside $V_j \cup \ldots \cup V_{j+m}$ of length at most s.*

Proof. Let $\gamma = (a = y_0, \ldots, y_1, \ldots, y_{k+1} = b) \subseteq V_j \cup \ldots \cup V_{j+m}$ change direction exactly in $y_1, \ldots y_k$. We can reduce γ by putting $z_0 = y_0$ and replacing for $i = 1, \ldots k$ inductively y_i by $z_i = z_{i-1} \vee y_{i+1}$ (or $z_i = z_{i-1} \wedge y_{i+1}$, respectively) filling the path with flags between z_i and z_{i+1}. The new path γ' has length at most s, is still in $V_j \cup \ldots \cup V_{j+m}$, and changes direction *at most* in $z_1, \ldots, z_k$. Note that if γ was not reduced, then the length must go down. We repeat the procedure with those vertices z_i in which γ' changes direction. This process stops after finitely many reductions with a reduced path of length at most s. ⊣

We need the following lemmas:

LEMMA 2.13. *If $a \in R(b)$, then any reduced path from a to b is a flag. If $a, b \in R_{\leq}(c)$ are connected by a reduced path γ, then $\gamma \subseteq R_{\leq}(c)$.*

PROOF. We prove both claims by induction on the length k of a reduced path from a to b. For $k = 1$ both statements are clear.

Now suppose that both statements are proved for paths of length at most $k-1$. Let $\gamma = (a = x_0, \ldots, x_k = b)$ be a reduced path with $a \in R_<(b)$. By induction assumption we have $(x_1, \ldots, x_k = b) \subset R_{\geq}(a)$, and hence $x_{k-1} \in R(a) \cap R(b)$. Hence again by induction $(a = x_0, x_1, \ldots, x_{k-1})$ is a flag. Since γ is reduced, it cannot change direction in x_{k-1}. It follows that γ is a flag.

For the second assertion let $\gamma = (a = x_0, \ldots, x_k = b)$ be a reduced path with $a, b \in R_{\leq}(c)$. If $x_i \in R_{\leq}(c)$ for some $1 \leq i \leq k-1$, then $\gamma \subset R_{\leq}(c)$ by induction assumption. So suppose there is no $1 \leq i \leq k-1$ with $x_i \in R_{\leq}(c)$. Then by Remark 2.3 γ contains some element whose level is higher than the level of c. Let $\gamma' = (a, y_1, \ldots, c, \ldots, b)$ be a path consisting of flags $(a, \ldots, c)$ and $(c, \ldots, b)$. So $\gamma \cap \gamma' = \{a, b\}$. Let $1 \leq i \leq k-1$ be such that the level of x_i is maximal. Let j_0, j_1 be minimal (maximal, respectively) such that $x_{j_0}, x_{j_1} \in R(x_i)$ and consider

$$\gamma'' = (x_{j_0}, \ldots, x_1, x_0 = a, y_1, \ldots, c \ldots, b = x_k, x_{k-1}, \ldots, x_{j_1}).$$

Again by Remark 2.3, the path γ'' contains in its interior some vertex $y \in R_<(x_i)$. By choice of j_0, j_1 we must have $y \in (a, \ldots, c \ldots, b)$. If y lies between a and c, then we have $a \in R_<(x_i)$ and hence $j_0 = 0$ and $(a = x_0, \ldots, x_i)$ is a flag by induction. But then a is not an interior point of γ''. Similarly if y lies between c and b, and this finishes the proof. ⊣

Note that if $a \in R_<(c), b \in R_>(c)$, then a reduced path from a to b is a flag, but need not contain c.

LEMMA 2.14. *Let $\gamma = (x_0, \ldots x_s)$ be a reduced path and let $(y_0 = x_0, \ldots y_k)$ be a flag. Let $m \leq s$ be minimal such that $y_k \notin R(x_m)$. Then for some $i \geq m-1$ the path $(y_k, \ldots, x_i, x_{i+1} \ldots, x_j, \ldots, x_s)$ is reduced and changes direction in x_i.*

PROOF. First suppose $(y_k, \ldots, x_0, \ldots, x_s)$ does not change direction in x_0 and, say, $y_k \in R_<(x_0)$. Let $j \geq m$ be minimal such that γ changes direction in x_j. Then $z = y_k \vee x_j \in R(x_{m-1})$. Hence we may assume that $z \in \{x_s \colon s = m-1, \ldots, j\}$ and the claim follows.

Now assume that $(y_k, \ldots, x_0, \ldots, x_s)$ changes direction in x_0 with $y_k \in R_>(x_0)$. Then $x_{m-1} = y_k \wedge x_m \in R_<(x_m)$ and $(y_k, \ldots, x_{m-1}, \ldots, x_s)$ is reduced and changes direction in $z = x_{m-1}$. ⊣

As in the theory of buildings we can show here that residues are *gated*, i.e., the following holds in any model M of T_n:

LEMMA 2.15. *Let M be a model of T_n. For all $a, c \in M$ with $a \notin R(c)$ there is a flag $f = (b_1, \ldots b_k) \subset R(c)$ such that any reduced path from a to c enters $R(c)$ via some element of f.*

PROOF. The proof is by induction on the rank of the pseudospace. The claim is clear if the rank is 1. Assume now that the claim is proved for rank less than n. For rank n, it suffices to prove the following:

If $\gamma_1 = (x_0 = a, \ldots x_s = c), \gamma_2 = (y_0 = a, \ldots y_t = c)$ are reduced paths and i, j are minimal with $x_i, y_j \in R(c)$, then $x_i = y_j$ or (x_i, y_j) is a flag.

Note that by the definition of a reduced path and by Lemma 2.13, x_i, y_j are exactly the last vertices where γ_1, γ_2 change direction.

We do induction on $\min\{i, j\}$, by symmetry we may assume $i \leq j$. If $i = 1$, then $a, c \in R_>(x_1)$ (up to duality) and $\gamma_1, \gamma_2 \in R_\geq(x_1)$ by Lemma 2.13. In particular, $y_j \in R \geq (x_1)$.

Now let $i > 1$. If $\gamma = (x_1, a, y_1, \dots y_t = c)$ is reduced we apply the induction hypothesis to the paths starting at x_1. So we may assume that $(x_1, a, y_1, \dots y_t = c)$ is not reduced.

First assume that γ changes direction in a. Let m be minimal such that γ_2 changes direction in y_m, so $m \leq j$. Since γ is not reduced, we have $x_1 \in R(y_m)$. So we may assume that $y_1 = x_1$ and apply the induction hypothesis to the paths starting from x_1.

If γ does not change direction in a, let us assume for fixing notation that $x_1 \in R_<(a)$. Let m be minimal with $x_1 \notin R(y_m)$ and such that γ_2 changes direction in y_m and put $z = x_1 \vee y_m$. Then $(x_1, \dots, z, \dots y_m, \dots, c)$ is a reduced path. If $j \geq m$, we may apply the induction hypothesis to x_1. If $j < m$, then $x_1 \in R_<(y_j)$. If $c \in R_>(y_j)$, then $x_1 \in R(c)$, contradicting $i > 1$. Hence $x_1, c \in R_<(y_j)$ and we finish by Lemma 2.13 as $(x_1, x_2, \dots c) \subseteq R_\leq(y_j)$. ⊣

As a first step toward showing that T_n is a complete theory, we show that any finite subset of a model of T_n is contained in a *nice* subset.

We call two reduced paths γ_1, γ_2 from a to b *equivalent*, $\gamma_1 \sim \gamma_2$ if they have the same set of vertices in which they change direction. Note that if $\gamma_1 \sim \gamma_2$, then γ_1 is reduced if and only γ_2 is.

Definition 2.16. Following [1] we call a subset A of a model M of T_n *nice* (in M) if A is in $\mathcal{K}_n$ and the following holds:

1. if $a, b \in A$ are connected (in M) by a reduced path γ of length k contained in $V_{i-m} \cup \dots \cup V_i$ in M there is an equivalent path γ' from a to b inside A.
2. If $a, a' \in A$, then $a \vee a' \in A$ if $a \vee a'$ exists in M.
3. If $a, a' \in A$, then $a \wedge a' \in A$ if $a \wedge a'$ exists in M.

Remark 2.17.

1. Note that (the proof of) Lemma 2.12 implies that if A is nice in a model M and $a, b \in A$ are connected by a path in M, then they are connected by a reduced path in A. Therefore conditions 4, 5, and 6 for graphs in $\mathcal{K}_n$ automatically hold for a nice set A.
2. If A is nice, $a, b \in A$, and $\gamma = (a = x_0, \dots, x_m = b)$ is a reduced path changing direction in $y_1, \dots, y_k$, then $y_1, \dots, y_k \in A$.
3. If A is nice, then also $A \cap (V_j \cup \dots \cup V_{j+m})$ is nice in the sense of the pseudospace $(V_j \cup \dots \cup V_{j+m})$ for all $j + m \leq n$. In particular, since E_i-paths between elements are unique, any E_i-path between elements of A lies entirely in A and if $a, b \in A$ are contained in a dense flag in M, they are contained in a dense flag of A. Furthermore, if A is nice and $b \in A$, then by Lemma 2.13 $R_\geq(b) \cap A$, $R_\leq(b) \cap A$ and hence $R(b) \cap A$ are nice. Note also that $R_>(b) \cap A$, $R_<(b) \cap A$ are nice in the sense of the pseudospace $R_<(b)$, $R_>(b)$, respectively.

LEMMA 2.18. *Let M be a model of T_n and let A, B be finite subsets of M with $A \leq B$. If $a, b \in A$ are connected by a path $\gamma \subset B$, then there is an equivalent path $\gamma' \subset A$. In particular, if B is nice, then so is A.*

PROOF. Write $B = \bigcup_{i<k} B_i$ with $B_0 = A$ and such that $B_{i-1} \leq B_i$ is a minimal strong extension for $i = 1, \ldots, k$. Let $\gamma = (a = x_0, \ldots, x_m = b)$ and let j be minimal with $\gamma \subseteq B_j$. Then B_j must be a strong extension of B_{j-1} of type 2, and we may replace the new flag of B_j by a flag in B_{j-1} to obtain an equivalent path in B_{j-1}. Continuing in this way we eventually find an equivalent path in A. ⊣

We will show that any finite set is contained in a nice strong finite set. To simplify inductive proofs we define a *pointed pseudospace (of dimension n)* as a free pseudospace V of dimension n together with a new vertex $x \in V_{n+1}$ or $x \in V_{-1}$ incident exactly with the elements of V_n (or of V_0). Thus, if V is a free pseudospace of dimension n and $b \in V_i$, then $R_{\leq}(b)$ is a pointed pseudospace of dimension $i-1$ and $R_{\geq}(b)$ is a pointed pseudospace of dimension $n-i-1$. We say that a subset of a pointed pseudospace is nice if it satisfies the conditions in Definition 2.16.

LEMMA 2.19. *Let M be a pseudospace or a pointed pseudospace. If $A \subset M$ is finite and nice in M and a is arbitrary, then there is a nice finite set B containing $A \cup \{a\}$ such that $A \leq B$.*

PROOF. The proof is by induction on the dimension n of the (pointed) pseudospace. The claim is clear for $n = 0$, so assume it has been proved for all (pointed) pseudospaces of dimension less than n. If M is a pointed pseudospace, let $x \in M$ be the additional point. Of course we may assume $a \notin A$. We may also assume that there is some reduced path $\gamma = (a = x_0, \ldots b)$ for some $b \in A$ and $\gamma \cap A = \{b\}$ as otherwise $A \cup \{a\}$ is nice. It therefore suffices to prove the claim for the case where $a \in V_i$ has a neighbor $b \in A$ of type V_{i+1} (the other case being dual to this one). If $b = x$ and any reduced path from a to an element of A passes through x, then $A \cup \{a\}$ is nice. Otherwise we may assume that $b \neq x$ so that we may apply the induction hypothesis to $R_{\leq}(b)$.

By Remark 2.17 we know that $R_{\leq}(b) \cap A$ is nice. By induction hypothesis we find a nice finite set $B \subseteq R_{\leq}(b)$ containing a such that $(R_{\leq}(b) \cap A) \leq B$. We claim that $A \leq A \cup B$ and $A \cup B$ is nice. To see this we write $B = \bigcup_{j<r} B_j$ as a union of minimal strong extensions over $B_0 = A \cap R_{\leq}(b)$ and show inductively that $A \cup B_{j-1} \leq A \cup B_j$ and $A \cup B_j$ is nice. Note that by Lemma 2.18 each B_j is nice.

Case I If $B_j = B_{j-1} \cup \{c\}$ is a strong extension of B_{j-1} of type 1 by some $c \in R_{<}(b)$, then since B_j is nice and $c \in R_{<}(b)$, we must have $c \in R_{<}(d)$ for a unique neighbor d of c with $d \in B_{j-1}$. Since $c \in R_{<}(b)$ (and B_j is nice in the sense of $R_{<}(b)$), we have $R_{<}(c) \cap (A \cup B_j) = R_{<}(c) \cap B_j = \emptyset$.

If c had another neighbor $d' \in A \setminus B_{j-1}$, then d, d' could not have the same level since otherwise by induction $c \in A \cup B_{j-1}$. So (d, c, d') must be a flag, contradicting our assumption. So c has no other neighbor in $A \cup B_{j-1}$, and hence $A \cup B_{j-1} \leq A \cup B_j$.

To see that $A \cup B_j$ is nice, note that for $x \in A \setminus B_j$ we must have $c \vee x = d \vee x$ since otherwise the path $(d, c, \ldots c \vee x, \ldots, x)$ is reduced and changes direction in c, yielding $c \in B_{j-1}$ by induction assumption. Hence Condition 2. holds.

To see that Condition 3 holds, note that for $x \in A$ we have $d \wedge x \in B_{j-1}$. If $\emptyset \neq c \wedge x \neq d \wedge x$, then the path $(c, \dots, c \wedge x, \dots d \wedge x)$ is reduced and changes direction in $c \wedge x$. Since B_j is nice we have $c \wedge x \in B_j$. So by assumption we must have $c = c \wedge x$, i.e., $c \in R_<(x)$.

Now suppose that c is connected to $x \in A \cup B_{j-1}$ by a reduced path $\gamma = (c = x_0, \dots, x_m = x)$. Consider the extension $\gamma' = (d, c = x_0, \dots, x_m)$.

If γ' did not change direction in c, then γ' could not be reduced since otherwise $x_i \in R_<(c) \cap B_{j-1}$ for some $0 < i < m$, a contradiction. So let i, j be minimal such that γ' changes direction in x_i and x_j. It is easy to see that we can reduce γ' to a path $(d, \dots, y = d \wedge x_j, \dots, x_j, x_{j+1}, \dots, x_m = x)$ changing direction in $y \in B_{j-1}$. But then $x_i = c \wedge y \in R_<(c) \cap B_j$ by Condition 3, again a contradiction.

Hence γ' changes direction in c. Then γ' cannot be reduced since otherwise by niceness of $A \cup B_{j-1}$ we have $c \in B_{j-1}$. We can reduce γ' to a path $(d, \dots, x_i, \dots, x_m = x)$ where i is minimal such that γ changes direction in x_i. Then the path $(c, d, \dots, x_i, \dots x_m = x)$ is equivalent to γ and by induction assumption we may replace the path $(d, \dots, x_i, \dots, x_m = x)$ by an equivalent one inside $A \cup B_{j-1}$.

Case II If B_j arises from B_{j-1} by a strong extension of type 2, then there are $b_1, b_2 \in B_{j-1}$ which are connected by a flag inside B_{j-1} and $B_j = B_{j-1} \cup \{x_1, \dots, x_k\}$ where $(b_1, x_1, \dots, x_k, b_2)$ is a flag, $b_1 \in R_<(b_2)$, say. If for some $i \in \{1, \dots, k\}$ the vertex x_i has a neighbor $d \in A \setminus B_{j-1}$, then since $d \notin R_<(b)$, we must have $x_i = d \wedge b_2 \in B_{j-1}$ by induction assumption. This is impossible since B_j is a minimal strong extension of B_{j-1}. Hence $A \cup B_j$ is a minimal strong extension of $A \setminus B_{j-1}$.

To see that $A \cup B_j$ is nice, let $c, d \in A \cup B_j$. If $c, d \in A \cup B_{j-1}$ or $c, d \in B_j$ there is nothing to show. So assume $c \in A \setminus B_j$ and $d = x_i \in B_j \setminus B_{j-1}$ for some $1 \leq i \leq k$. For Condition 2, we claim that $c \vee d = c \vee b_2$. Suppose to the contrary that $c \vee d \neq c \vee b_2$ and consider the path $(b_2, \dots, d, \dots, d \vee c, \dots, c)$. If this path is reduced, then since it changes direction in d, by induction assumption we have $d \in A \cup B_{j-1}$ and hence $d \in B_{j-1}$, a contradiction. Hence we find a reduced path $(b_2, \dots, y \dots, d \vee c, \dots, c)$ with $y = b_2 \wedge (d \vee c) \in B_{j-1}$. Since B_j is a minimal strong extension of B_{j-1}, we must have $y = b_2$ and hence $b_2 \in R_<(d \vee c)$. Therefore $b_2 \vee c = d \vee d$, contradicting the assumption.

Similarly, for Condition 3, we claim that $c \wedge d = c \wedge b_1$, the argument being exactly dual to the one for Condition 2.

For Condition 1, assume now that c, d are connected by a reduced path $\gamma = (d = y_0, \dots, y_\ell = c)$ and extend γ by the flag $(b_2, \dots, x_i = d = y_0)$ to a path γ'. Then since $c \notin R_<(b_2) \subseteq R_<(b)$, there is a minimal t such that $y_t \notin R_<(b_2)$.

By Lemma 2.14 there is a reduced path from b_2 to c changing direction in y_s for some $s \geq t - 1$. Then $y_s \in A \cup B_{j-1}$, and so $y_s \in R_<(b)$ implies $y_s \in B_{j-1}$ by induction assumption. By induction assumption and niceness of B_j we find inside $A \cup B_j$ a path from d to c equivalent to γ. ⊣

Remark 2.20. The construction shows that for a path $\gamma = (x_0, \dots, x_k)$ changing direction in $y_1, \dots, y_k$ there is a nice finite set $A \supset \gamma$ with $A \subseteq \bigcup_{i=1,\dots k} R_\delta(y_i)$ where $\delta \in \{\leq, \geq\}$ depending on the change of direction in y_i.

Corollary 2.21.

1. *If A is a finite subset of a model M of T_n, there is a nice finite set B in M containing A.*
2. *For any two vertices a and b, up to equivalence there are only finitely many reduced paths from a to b.*
3. *Reduced paths $\gamma = (a = x_0, \ldots, x_k = b), \gamma' = (a = y_0, \ldots, y_k = b)$ from a to b are equivalent if (and only if) x_i, y_i have the same level for all $i = 0, \ldots, k$.*

Proof. Part 1. is clear and Part 2. follows directly from 1. The proof of the nontrivial direction for 3. is by induction on the number of vertices in which γ (and γ') change direction. If they change direction exactly once, the claim is clear. By Lemma 2.15 the last change of direction of γ and γ' is in the same element of $R(b)$. Hence the claim follows from the induction hypothesis. ⊣

We also note the following corollary:

Corollary 2.22. *If $A_0 \subseteq M_n$ is a nice finite set, then we can write $M_n = \bigcup_{i<\omega} A_i$ with $A_i \leq A_{i+1}$.*

We say that a model M of T_n is $\mathcal{K}_n$-*saturated* if for all nice finite sets $A \subset M$ and strong extensions C of A with $C \in \mathcal{K}_n$ there is a nice embedding of C into M fixing A elementwise. The following lemma shows that M_n is $\mathcal{K}_n$-saturated:

Lemma 2.23. *Write $M_n = \bigcup_{i<\omega} A_i$ with $A_i \in \mathcal{K}_n$ and $A_i \leq A_{i+1}$. Then each A_i is nice in M_n.*

Proof. Fix i and let $a, b \in A_i$. If $\gamma = (a = x_0, \ldots, x_k = b)$ is a path, then $\gamma \subseteq A_{i+s}$ for some $s > i$. By Lemma 2.18 there is an equivalent path inside A_i, so A_i is nice. ⊣

Lemma 2.24. *An L_n-structure M is an ω-saturated model of T_n if and only if M is $\mathcal{K}_n$-saturated.*

Proof. Let M be an ω-saturated model of T_n. To show that M is $\mathcal{K}_n$-saturated, let $A \subset M$ be a nice finite set and $A \leq B \in \mathcal{K}_n$. By induction we may assume that B is a minimal strong extension of A. By ω-saturation it is easy to see that B can be nicely embedded over A into M. Conversely, assume that M is $\mathcal{K}_n$-saturated. Since any finite subset A of M is contained in a nice finite set $B \subseteq M$ we see that M is back-and-forth equivalent to M_n and so is a model of T_n. Choose an ω-saturated $M' \equiv M$. Then by the above M' is $\mathcal{K}_n$-saturated. So M' and M are also back-and-forth equivalent, which implies that M is ω-saturated. ⊣

Corollary 2.25. *The theory T_n is complete.*

Proof. Let M be a model of T_n. In order to show that M is elementarily equivalent to M_n choose an ω-saturated $M' \equiv M$. By Lemma 2.24, M' is $\mathcal{K}_n$-saturated. Now M' and M_n are back-and-forth equivalent and therefore elementarily equivalent. ⊣

Remark 2.26. Note that this implies also that for nice finite sets A the quantifier free type of A already determines the complete type of A.

We will see in Section 4 that T_n is the theory of the building of type $A_{\infty,n+1}$ with infinite valencies.

COROLLARY 2.27. *The algebraic closure* $\mathrm{acl}(A)$ *of a finite set* A *is the intersection of all nice sets containing* A.

PROOF. Clearly, the intersection of all nice sets is contained in $\mathrm{acl}(A)$ by Lemma 2.19. For the converse, assume B is a nice set containing A and $x \notin B$. Let $D \supset B \cup \{x\}$ be a nice set. Since M_n is $\mathcal{K}_n$-saturated and the free amalgam of any finite number of copies of D over B is again in $\mathcal{K}_n$, we can find infinitely many copies of D over B in $\mathcal{K}_n$. ⊣

Using the fact that M_n is ω-saturated we can now give an explicit description of the algebraic closure:

PROPOSITION 2.28. *A vertex* $c \neq a, b$ *is in* $\mathrm{acl}(ab)$ *if and only if there is a reduced path from* a *to* b *changing direction in* c. *Hence* $\mathrm{acl}(ab) = \{a, b\}$ *if and only if* a, b *is a flag or* a *and* b *are not connected.*

PROOF. By Corollary 2.27 any c for which there is a reduced path from a to b that changes direction in c is in $\mathrm{acl}(ab)$.

Now let $c \neq a, b$ and suppose that there is no reduced path between a and b changing direction in c. If there is no (reduced) path from a to b, the set $\{a, b\}$ is already nice. So let $\gamma = (a = x_0, \dots, \dots x_s = b)$ be a reduced path. We may assume $c \notin \gamma$. We construct a nice set containing a, b but not c. We may also assume that $|\gamma \cap R(c)|$ is minimal. For rank $n = 1$ this is easy. So we may assume that we have proved the claim for all pseudospaces of rank less than n. Also, if $R(c)$ does not intersect γ, then Remark 2.20 shows that we can find a nice set containing γ but not c. So let k be maximal such that for some $i \leq s$ we have $x_i, x_{i+k} \in R(c)$. Note that γ must change direction in x_i and x_{i+k} as otherwise we could replace x_i, x_{i+k} by other elements to minimize $|\gamma \cap R(c)|$. Suppose that $c \in V_m$.

First suppose that $k > 0$ and both $x_i, x_{i+k} \in R_<(c)$ (or by symmetry both in $R_>(c)$) so that $R_<(c) \cap \gamma = (x_i, \dots, x_{i+k})$ by Lemma 2.13. Note that this implies that x_i, x_{i+k} have level at most $m - 2$. Then γ must change direction at some place between x_i and x_{i+k} as this path cannot be a flag. If $x_i \vee x_{i+k} = c$ then we claim that we can obtain a reduced path from a to b changing direction in c: namely if a is the last place where γ changes direction before x_i, then $z_1 = a \wedge c \in R_<(a)$. Similarly if b is the first place where γ changes direction after x_{i+k}, then $z_2 = b \wedge c \in R_<(b)$. Replacing in γ the path $(a, \dots, b)$ by $(a, \dots, z_1, \dots, c, \dots, z_2, \dots, b)$, we obtain a path, which is easily seen to be reduced, changing direction in c.

Hence $z = x_i \vee x_{i+k} \in R_<(c)$. We may assume that the path from x_i to x_{i+k} consists of flags $(x_i, \dots, z)$ and $(z, \dots, x_{i+k})$. By induction assumption and the fact that $z \in R_<(c)$ we find an absolutely nice set $D_1 \subseteq R_<(c)$ containing $(x_i, \dots, x_{i+k})$ (and not containing c). So $D_1 \subseteq V_0 \cup \dots \cup V_{m-1}$. Since $x_{i-1}, x_{i+k+1} \in V_0 \cup \dots \cup V_{m-1}$ we can extend D_1 to a nice set $D_2 \subset V_0 \cup \dots \cup V_{m-1}$ containing x_{i-1}, x_{i+k+1}. We extend D_2 to a nice set D containing $(x_0, \dots x_{i-2})$ and $(x_{i+k+2}, \dots, x_s)$ as described in Lemma 2.19. Since $c \notin R(x_j)$, $j = 0, \dots, i-2, i+k+2, \dots, s$ we have $c \notin D$.

Now suppose that $k = 0$ and $x_i \in R_<(c)$. Let a', b' be the last place before (and the first place after) x_i where γ changes direction, so $a', b' \in R_>(x_i)$.

Claim: There is no reduced path from a' to b' changing direction in c.

Suppose otherwise and let $\gamma' = (a' = y_0, \dots, y_s = b')$. Then c cannot be the first or last place where γ' changes direction since $k = 0$. If $y_1 \in R_<(a')$, $y_{s-1} \in R_<(b')$, then replacing $(a', \dots, x_i, \dots, b')$ by γ' in γ yields a reduced path changing direction in c, contradicting the assumption. If $y_1 \in R_>(a')$ let $d_0 = a', d_1, d_2, \ldots \in \gamma'$ an enumeration of the places where γ' changes direction and let a_1 be the last place before a' where γ changes direction. Put $z_1 = a_1 \vee d_2 \in R_\leq(d_1)$, so $z_1 \wedge d_3 = d_2$. We can now reduce the path starting from z_1 from right to left as in Lemma 2.12. If necessary we can do the same on the right end of γ' and thus obtain a reduced path from a to b changing direction in c, a contradiction proving the claim.

By the claim we can apply the induction assumption to $R_>(x_i)$. So let $D \subset R_>(x_i)$ be a nice set in the sense of $R_>(x_i)$ containing x_{i-1}, x_{i+1}, but not containing c. Then $D_1 = D \cup \{x_i\}$ is nice in the sense of the $V_0 \cup \ldots \cup V_n$ and we can extend D_1 to a nice set D_2 as in Lemma 2.19. Since $c \notin R(x_j), j = 0, \ldots, i-2, i+2, \ldots, s$ we have $c \notin D_2$.

Finally suppose that $x_i \in R_<(c), x_{i+k} \in R_>(c)$ (the other case is similar), so that $(x_i, \ldots, x_{i+k})$ is a flag and hence a nice set. If $x_{i+k} \in V_k$ for $k < n$, then we can extend the nice set $\{x_i, \ldots, x_{i+k}\}$ by induction assumption to a nice set containing x_{i-1}, x_{i+k+1}, but not containing c and we continue as in Lemma 2.19. If $x_{i+k} \in V_n$ and every reduced path from a to b passes through x_{i+k}, then we can use the induction assumption to construct a nice set in $V_0 \cup \ldots \cup V_{n-1}$ containing $(x_{i-1}, \ldots, x_{i+k-1})$ and not c. This can be extended to a nice set in $V_0 \cup \ldots \cup V_n$ by adding the necessary elements in V_n. We can also construct a nice set containing $(x_{i+k+1}, \ldots b)$. The union of these two sets together with x_{i+k} will be a nice set since every path from one of these sets to the other set has to pass through x_{i+k}.

Now suppose that there is a path from x_{i+k-1} to x_{i+k+1} not passing through x_{i+k}. Since $x_{i+k-1}, x_{i+k+1} \in R_<(x_{i+k})$, by ($\Sigma$4) there is such a path $\gamma' = (x_{i+k-1} = y_0, \ldots, y_t = x_{i+k+1})$ in $R_<(x_{i+k})$. We can now apply the induction assumption to find a nice set in $V_0 \cup \ldots \cup V_{n-1}$ containing $(x_{i-1}, x_i, \ldots x_{i+k+1})$, but not containing c. We extend this to a nice set D in $V_0 \cup \ldots \cup V_n$ by adding the necessary elements of V_n and continue as in Lemma 2.19. ⊣

Proposition 2.29. *For any set A we have* $\mathrm{acl}(A) = \mathrm{dcl}(A)$.

Proof. It suffices to prove the proposition for finite sets A. If $|A| = 1$, we have $\mathrm{acl}(A) = A$ by ω-saturation of M_n. Now assume that the statement is true for A and let $a \notin \mathrm{acl}(A) = \mathrm{dcl}(A)$. By Corollary 2.21 it suffices to prove the following:

Claim 2.30. We have $c \in \mathrm{acl}(aA) \setminus \mathrm{acl}(A)$ if and only if there is a reduced path $\gamma = (a = x_0, \ldots, x_s = b)$ for some $b \in \mathrm{acl}(A)$ changing direction in c.

Clearly, if there is a reduced path $\gamma = (a = x_0, \ldots, x_s = b)$ for some $b \in \mathrm{acl}(A)$ changing direction in c, then $c \in \mathrm{dcl}(aA)$ by Proposition 2.31 and Corollary 2.21. For the converse, it suffices to construct a nice set D containing $A \cup \{a\}$ but not c. So let D_0 be a nice finite set containing A and not c and let $\gamma = (a = x_0, \ldots, x_m = b)$ be a reduced path with $b \in \mathrm{acl}(A)$. Exactly as in Proposition 2.28, we construct a nice set $D \geq D_0$ containing a but not c. ⊣

In Section 4 we will see that in the prime model the algebraic closure will be described by reduced words in the Coxeter group associated to the building.

We next show that algebraically closed sets are *weakly gated* in the following sense:

Proposition 2.31. *For any set A and $a \notin$* $\mathrm{acl}(A)$, *there is a flag $C \in$* $\mathrm{acl}(A)$ *such that for any $b \in$* $\mathrm{acl}(A)$ *and any reduced path from a to b there is an equivalent one* *entering* acl(A) *through one of the elements of* C *and this is the last vertex where the path changes direction.*

The proposition follows from the following lemma:

LEMMA 2.32. *Let* $b_1, b_2 \in \operatorname{acl}(A), a \notin \operatorname{acl}(A)$ *and let* $\gamma_1 = (a = x_0, \dots, x_s = b_1)$ *and* $\gamma_2 = (a = y_0, \dots, y_t = b_2)$ *be reduced paths. Assume that* γ_1, γ_2 *are chosen in their equivalence classes such that* i, j *are minimal with* $x_i, y_j \in \operatorname{acl}(A)$. *Then* $x_i \in R(y_j)$.

PROOF. Let γ denote the extension of γ_2 by $(x_i, \dots, x_1, a)$. If γ is reduced, then $(x_i, \dots, a, y_1, \dots y_j)$ is a flag by minimality of i, j and we are done. Now assume that γ is not reduced.

Case I: Suppose that γ does not change direction in a. To fix notation let $x_1 \in R_<(a)$. We do induction on $\min\{i, j\}$. By symmetry we may assume $i \leq j$. First assume $i = 1$. Let m be minimal such that $x_1 \notin R(y_m)$. Note that in this case γ_2 must change direction in y_{m-1}. If $j < m$ we are done. Otherwise as in Remark 2.14 we obtain a reduced path $(x_1, \dots, y_{m-1}, y_m, \dots, y_t)$ containing a flag from x_1 to y_{m-1}. Then $y_{m-1} \in \operatorname{acl}(A)$, contradicting the minimality of j.

So now assume that $i > 1$ and consider the path $\gamma' = (x_1, a, y_1, \dots, y_t)$. If γ' is reduced we may apply the induction hypothesis to γ' and the path from x_1 to b_1. Otherwise we reduce γ' as in Remark 2.14: let m be minimal such that $x_1 \notin R(y_m)$, so $y_{m-1} = x_1 \vee y_m$. If $j > m$ we obtain again a reduced path $(x_1, \dots, y_{m-1}, y_m, \dots, y_t)$ containing a flag from x_1 to y_{m-1} and use the induction hypothesis on this path and γ_1 starting from x_1. If $j < m$ we have $x_1 \in R(y_j)$. Let k be minimal such that $x_k \notin R(y_j)$. If $k > i$ we are done. Otherwise we obtain a reduced path $(y_{j-1}, \dots, x_{k-1}, x_k, \dots, x_s = b_1)$ and apply the induction hypothesis to this and γ_2 starting from y_{j-1}.

Case II: Suppose that γ changes direction in a, say $x_1 \in R_>(a)$. We do induction on $\min\{i, j\}$. By symmetry we may assume $i \leq j$. Let m be minimal such that γ_2 changes direction in y_m.

We start with $i = 1$. Since γ is not reduced, we have $x_1 \in R(y_m)$ since we must have $x_1 \wedge y_m = x_1$. Since $a \notin \operatorname{acl}(A)$, the path $(x_1, a, y_1, \dots y_j)$ is not reduced as it changes direction in a. But $(a, y_1, \dots y_j)$ is reduced and we have $a, y_m \in R(x_1)$. Since x_1, y_1 have the same level, we conclude that $x_1 = y_1 \in \operatorname{acl}(A)$ and we finish.

Now assume $i > 1$ and consider the path $\gamma' = (x_1, a, y_1, \dots y_t)$. If γ' is reduced, we may apply the induction hypothesis to the paths starting at x_1. Otherwise we have as before $x_1 \in R(y_m)$. If $x_1 \notin R(y_j)$, then by the proof of Lemma 2.14 we see that the path $(y_j, \dots, y_1, a, x_1, \dots, x_i)$ is reduced and changes direction in a implying that $a \in \operatorname{acl}(A)$. Hence $x_1 \in R(y_j)$. In γ_2 we may therefore replace $(a, y_1, \dots, y_j)$ by $(a, x_1, \dots, y_j)$ and apply the induction hypothesis to γ_1, γ_2 starting from $x_1 = y_1$. ⊣

COROLLARY 2.33. *For any type* $\operatorname{tp}(a/A)$, *if* C *is a flag in* $\operatorname{acl}(A)$ *such that for any* $b \in \operatorname{acl}(A)$ *and any reduced path from* a *to* b *there is an equivalent one entering* $\operatorname{acl}(A)$ *through one of the elements of* C *and if no proper subset of* C *has this property, then* C *is uniquely determined.*

PROOF. Let $D \subset \operatorname{acl}(A)$ be another flag with the same property and let $d \in D \setminus C$. If $\gamma = (a, \dots, d)$ is a reduced path, we may assume γ to enter $\operatorname{acl}(A)$ via an element $c \in C \subset \operatorname{acl}(A)$. Thus we may assume that $\gamma = (a, \dots, c, \dots, d)$. Now by assumption there is an equivalent path from a to c entering $\operatorname{acl}(A)$ via an element d' of D. So we may assume that $\gamma = (a, \dots, d', \dots, c, \dots, d)$ is a reduced path. Thus any reduced path from a to an element of $\operatorname{acl}(A)$ which contains d is equivalent to one that enters $\operatorname{acl}(A)$ via some element of $D \setminus \{d\}$, contradicting the minimality of D. ⊣

The flag C is called the *projection* from a to A and we write $C = \operatorname{proj}(a/A)$. Note that $\operatorname{proj}(a/A) = \emptyset$ if and only if a is not connected to any vertex of $\operatorname{acl}(A)$.

It is now easy to show the following:

THEOREM 2.34. *The theory T_n is ω-stable.*

PROOF. Let M be a countable model and let $\bar{d}$ be a tuple from $\overline{M}$. Let $C \in M$ be the finite set of projections from $\bar{d}$ to M. Then the type $\operatorname{tp}(\bar{d}/M)$ is determined by $\operatorname{tp}(\bar{d}/C)$. By Lemma 2.19, $\bar{d} \cup C$ is contained in a nice finite subset of M_n and for such subsets the quantifier-free type determines the type by Remark 2.26. Hence there are only countably many types over a countable model. ⊣

In fact, using the characterization of forking independence in stable theories (see [6] Ch. 8) it is easy to see directly without counting types that T_n is superstable. Note that since for any set A we have $\operatorname{acl}(A) = \operatorname{dcl}(A)$ this is an example of a stationary independence relation as defined in [7].

THEOREM 2.35. *Forking independence in models of T_n is given by*

$$A \underset{C}{\perp\!\!\!\perp} B$$

if and only if for all $a \in \operatorname{acl}(AC), b \in \operatorname{acl}(BC)$ and any reduced path from a to b there is an equivalent path passing through an element of $\operatorname{acl}(C)$.

PROOF. It is easy to see that this notion of independence satisfies the characterizing properties of forking in stable theories (see [6] Ch. 8) and hence agrees with the usual one. The existence of nonforking extensions follows from the construction of M_n as a Hrushovski limit. Since we have just seen that for any type $\operatorname{tp}(a/A)$ there is a finite set $A_0 \subseteq \operatorname{acl}(A)$ such that $a \perp\!\!\!\perp_{A_0} A$ this shows directly (without counting types) that T_n is superstable. ⊣

For convenience we also state the following special case:

COROLLARY 2.36. *The vertex a is independent from A over C if* $\operatorname{proj}(a/AC) \subseteq \operatorname{acl}(C)$, *i.e., if for any $b \in \operatorname{acl}(AC)$ connected to a and any reduced path from a to b there is an equivalent path passing through an element of* $\operatorname{acl}(C)$. *In particular, a is independent from A over $\emptyset$ if and only if a is not connected to any vertex of* $\operatorname{acl}(A)$ *by a path.*

This characterization of forking now directly implies the following:

COROLLARY 2.37. *The free pseudospace has weak elimination of imaginaries, i.e., any type has a canonical basis consisting of a finite set of real elements: for any vertex a in a saturated model $\overline{M}$ of T_n, the projection* $\operatorname{proj}(a/M)$ *of a onto a saturated elementary submodel M is a canonical basis for the type of a over M.*

§3. Ampleness.

We now recall the definition of a theory being n-ample given by Pillay and Evans in [2, 5], in slightly more symmetrized form:

DEFINITION 3.1. A theory T is called n-ample if (possibly after naming parameters) there are tuples $a_0, \ldots a_n$ in M such that the following holds:

1. for $i = 0, \dots n-1$ we have
$$\operatorname{acl}(a_0, \dots, a_{i-1}, a_i) \cap \operatorname{acl}(a_0, \dots a_{i-1}, a_{i+1}) = \operatorname{acl}(a_0, \dots a_{i-1});$$
2. $a_i \not\perp a_j$ for $i, j = 1, \dots n$, and
3. $a_0 \dots a_{i-1} \perp_{a_i} a_{i+1}, \dots, a_n$ for $i = 1, \dots n-1$.

Remark 3.2. If $a_0, \dots a_n$ witness n-ampleness over some parameters A, then for any $i = 1, \dots n$ we have
$$\operatorname{acl}(a_0, a_i) \cap \operatorname{acl}(a_0, a_{i-1}) \subseteq \operatorname{acl}(a_0 A).$$

Theorem 3.3. *The theory T_n is n-ample, but not $n+1$-ample.*

Proof. Any maximal flag $(x_0, \dots x_n)$ in M_n is a witness for n-ampleness. This follows immediately from the description of acl in Proposition 2.28 and of forking in Corollary 2.36.

To see that the free pseudospace of dimension n is not $n+1$-ample, suppose toward a contradiction that $A_0, \dots A_{n+1}$ are witnesses for T_n being $n+1$-ample over some set of parameters A. We have
$$A_{n+1} \underset{A}{\not\perp} A_0,$$
$$A_0 \dots A_{i-1} \underset{AA_i}{\perp} A_{i+1} \dots A_{n+1}, i = 0, \dots n.$$

By the first condition there are vertices $a_0 \in \operatorname{acl}(A_0 A) \setminus \operatorname{acl}(A)$ and $a_{n+1} \in \operatorname{acl}(A_{n+1} A) \setminus \operatorname{acl}(A)$ which are in the same connected component and not independent over A. We may choose a_0 and a_{n+1} at minimal distance with this property, i.e., in such a way that no reduced path from a_0 to a_{n+1} contains an element $b \in \operatorname{acl}(A_0 A)$ with $b \not\perp_A a_{n+1}$.

Since $a_0 \perp_{A_n A} a_{n+1}$ and $a_0 \not\perp_A a_{n+1}$ by the characterization of independence in Corollary 2.36 there is a flag $C_n \subset \operatorname{acl}(A_n A)$, $C_n \not\subset \operatorname{acl}(A)$ such that for any reduced path from a_0 to a_{n+1} there is an equivalent one passing through an element of C_n, so $a_0 \perp_{C_n} a_{n+1}$. Clearly, we may assume that C_n is minimal with this property. Let γ be a reduced path from a_0 to a_{n+1} not equivalent to a path containing an element of $\operatorname{acl}(A)$. Let $a_n \in C_n$ be such that γ or an equivalent path passes through a_n, then $a_0 \not\perp_A a_n$. If γ changes direction in some b between a_0 and a_n or in a_n, then by the previous remark $b \in \operatorname{acl}(a_0, a_{n+1}) \cap \operatorname{acl}(a_0, a_n) \subseteq \operatorname{acl}(A_0 A)$ and $b \not\perp_A a_{n+1}$, contradicting the choice of a_0. Therefore (a_0, a_n) form a flag with $a_n \notin V_n \cup V_0$ as otherwise γ changes direction in a_n. Now there is some flag $C_{n-1} \in \operatorname{acl}(A_{n-1} A), C_{n-1} \not\subset \operatorname{acl}(A)$ such that $a_0 \perp_{C_{n-1}} a_n$. Inductively, we find $a_i \in \operatorname{acl}(A_i A)$ such that $(a_0, a_i, \dots a_n)$ is a flag for $i = 1, \dots n-1$. This is impossible if $a_n \notin V_0 \cup V_n$. ⊣

The proof shows that in fact the following stronger ampleness result holds:

Corollary 3.4. *If $a_0, \dots a_n$ are witnesses for T_n being n-ample, then there are vertices $b_i \in \operatorname{acl}(a_i)$ such that $(b_0, \dots b_n)$ is a flag.*

The following was pointed out by Itay Ben Yacov:

Remark 3.5. If we let T_ω denote the theory of ω-colored graphs with vertices of type $\bigcup_{i<\omega} V_i$ such that for all $i, j \in \omega$ the restriction to $V_i \cup \dots V_{i+j}$ is a model of T_j, we see that T_ω is ω-stable, n-ample for all $n < \omega$ and does not interpret an infinite group.

§4. Buildings and the prime model of T_n. Let M be a model of T_n. We say that maximal flags ζ_1, ζ_2 are *totally connected* if for $i = 0, \dots n-1$ the V_i-vertices of ζ_1, ζ_2 are E_i- and E_{i+1}-connected (whenever this makes sense). Given a flag ζ in M we let $M^0(\zeta)$ be its totally connected component. Since M_n is homogeneous for maximal flags, all totally connected components of M_n are isomorphic.

The uniqueness part of Proposition 4.4 follows directly from the following theorem and Proposition 5.1 of [4] which states that this type of building is uniquely determined by its associated Coxeter group and the cardinality of the residues.

THEOREM 4.1. *M_n^0 is a building of type $A_{\infty,n+1}$ all of whose residues have cardinality $\aleph_0$.*

Recall the following definitions (see, e.g., [3]). Let W be the Coxeter group

$$W = \langle t_0, \dots t_n \colon t_i^2 = (t_i t_k)^2 = 1, i,k = 0 \dots n, |k-i| \geq 2 \rangle,$$

whose associated diagram we call $A_{\infty,n+1}$.

DEFINITION 4.2. A building of type $A_{\infty,n+1}$ is a set Δ with a *Weyl distance function* $\delta : \Delta^2 \to W$ such that the following two axioms hold:

1. For each $s \in S := \{t_i, i = 0, \dots n\}$, the relation $x \sim_s y$ defined by $\delta(x,y) \in \{1, s\}$ is an equivalence relation on Δ and each equivalence class of $\sim_s$ has at least 2 elements.
2. Let $w = r_1 r_2 \dots r_k$ be a shortest representation of $w \in W$ with $r_i \in S$ and let $x, y \in \Delta$. Then $\delta(x,y) = w$ if and only if there exists a sequence of elements $x, x_0, x_1, \dots, x_k = y$ in Δ with $x_{i-1} \neq x_i$ and $\delta(x_{i-1}, x_i) = r_i$ for $i = 1, \dots, k$.

A sequence as in 2. is called a *gallery of type* $(r_1, r_2, \dots, r_k)$. The gallery is called reduced if the word $w = r_1 r_2, \dots, r_k$ is reduced, i.e., a shortest representation of w.

We now show how to consider M_n^0 as a building of type $A_{\infty,n+1}$.

PROOF. (of Theorem 4.1) We extend the set of edges of the $n+1$-colored graph M_n^0 by putting edges between any two vertices that are incident in the sense of Definition 2.1.2.1. In this way, flags of M_n^0 correspond to a complete subgraph of this extended graph, which thus forms a simplicial complex. A maximal simplex consists of $n+1$ vertices each of a different type V_i. (Such a simplex is called a *chamber*.) Let Δ be the set of maximal simplices in this graph. Define $\delta : \Delta^2 \to W$ as follows:

Put $\delta(x,y) = t_i$ if and only if the flags x and y differ exactly in the vertex of type V_i. Extend this by putting $\delta(x,y) = w$ for a reduced word $w = r_1 r_2 \dots r_k$ if and only if there exists a sequence of elements $x = x_0, x_1, \dots, x_k = y$ in Δ with $x_{i-1} \neq x_i$ and $\delta(x_{i-1}, x_i) = r_i$ for $i = 1, \dots, k$.

Clearly, with this definition of δ, the set Δ satisfies the first condition of Definition 4.2. In fact, for all $s \in S$ every equivalence class $\sim_s$ has cardinality $\aleph_0$. We still need to show that δ is well-defined, i.e., we have to show the following for any $x, y \in \Delta$: if there are reduced galleries $x_0 = x, x_1, \dots, x_k = y$ and $y_0 = x, y_1, \dots, y_m = y$ of type $(r_1, r_2, \dots, r_k)$ and $(s_1, \dots s_m)$, respectively, then

in W we have $r_1 r_2 \dots r_k = s_1 \dots s_m$. Equivalently, we will show the following, which completes the proof of Theorem 4.1:
Claim: There is no reduced gallery $a_0, a_1, \dots, a_k = a_0$ for $k > 0$ in M_n^0.

Proof of Claim. Suppose otherwise. Let $a_0, a_1, \dots, a_k = a_0$ be a reduced gallery of type $(r_1, \dots r_k)$ for some $k > 0$. Note that the flags a_{i-1} and a_i contain the same vertex of type V_j as long as $r_i \neq t_j$.

Now consider the sequence of vertices of type V_n and V_{n-1} occurring in this gallery. Since $V_n \cup V_{n-1}$ contains no cycles, the sequence of vertices of type V_n and V_{n-1} occurring in this gallery will be of the form

$$(x_1, y_1, x_2, y_2, \dots, x_i, y_i, x_i, y_{i-1}, \dots y_1, x_1) \tag{1}$$

with $x_i \in V_n, y_i \in V_{n-1}$ and x_i a neighbor of y_i and y_{i-1} in the original graph. This implies that at some place in the gallery type there are two occurrences of t_n which are not separated by an occurrence of t_{n-1} (or conversely). Since t_n commutes with all t_i for $i \neq n-1$ and the word $r_1 \dots r_k$ is reduced, the first case cannot happen. Hence there are two occurrences of t_{n-1} which are not separated by an occurrence of t_n, say $r_j, r_{j+m} = t_{n-1}$ with $r_{j+1}, \dots, r_{j+m-1} \neq t_n$.

We now consider the gallery $a_j, \dots a_{j+m}$ of type $(r_j = t_{n-1}, r_{j+1}, \dots, r_{j+m} = t_{n-1})$. Notice that by (1), the flags a_j and a_{j+m} have the same V_n and the same V_{n-1} vertex. Since M_n^0 does not contain any E_{n-1}-cycles, the sequence of V_{n-1}- and V_{n-2}-vertices appearing in this sequence must again be of the same form as in (1). Exactly as before we find two occurrences[2] of t_{n-2} in the gallery type of $a_j, \dots a_{j+m}$ which are not separated by an occurrence of t_{n-1}. Continuing in this way, we eventually find two occurrences of t_1 which are not separated by any t_i. Since $t_1^2 = 1$ this contradicts the assumption that the gallery be reduced. ⊣

The proof shows in fact the following:

Corollary 4.3. *A model of T_n is a building of type $A_{\infty,n+1}$ if and only if it is E_i-connected for all i and every vertex is contained in a maximal flag.*

Proposition 4.4. *$M^0(\zeta)$ is a model of T_n. Furthermore M_n^0 is the unique countable model of T_n which is E_i-connected for $i = 1, \dots n$ and such that every vertex is contained in a maximal flag.*

(The corresponding Remark 3.6 of [1] uses Lemma 3.2, which is not correct as phrased there: M_n^0 and $M_n^0 \cup \{a\}$ with a an isolated point are not isomorphic, but satisfy the assumptions of Remark 3.6.)

Theorem 4.5. *The building M_n^0 is the prime model of T_n.*

Proof. To see that M_n^0 is the prime model of T_n note that for any flags $C_1, C_2 \in M_n^0$ and gallery $C_1 = x_0, \dots, x_k = C_2$ the set of vertices occurring in this gallery is E_i-connected for all i. Hence by Remark 2.26 its type is determined by the quantifier-free type.

Thus, given a maximal flag M in any model of T_n and a maximal flag c_0 of M_n^0 we can embed M_n^0 into M by moving along the galleries of M_n^0. ⊣

[2]If t_{n-2} does not occur in the type of the gallery, this would contradict the assumption that the type is reduced since t_{n-1} commutes with all t_i for $i \neq n, n-2$.

§5. Ranks and types. Recall that for vertices $x, y \in M_n^0$ with $x \in V_i, y \in V_j$ the *Weyl-distance* $\delta(x, y)$ equals $w \in W$ if there are flags C_1, C_2 containing x, y, respectively, with $\delta(C_1, C_2) = w'$ and such that w is the shortest representative of the double coset $\langle t_k : k \neq i \rangle w' \langle t_k : k \neq j \rangle$ (where as usual $\langle X \rangle$ denotes the subgroup of W generated by X).

Note that the Weyl distance between two vertices describes exactly the reduced paths between these vertices. Therefore we have the following:

PROPOSITION 5.1. *The theory T_n has quantifier elimination in a language containing predicates $\delta_w^{i,j}$ for Weyl distances between vertices of type V_i and of type V_j.*

PROOF. Since the predicates $\delta_w^{i,j}$ describe the reduced paths between vertices, in this language two tuples have the same quantifier-free type if and only if they are contained in isomorphic nice sets. For nice sets the quantifier-free type determines the type, whence the claim. $\dashv$

Using the description of forking given in Theorem 2.35 it is easy to give a list of regular types such that any nonalgebraic type is nonorthogonal to one of these. This is entirely similar to the list given in [1] and we omit the details. It is also clear from this description of forking that the geometry on these types is trivial.

For any small set A in a large saturated model we have the following kinds of regular types:

(I) $\mathrm{tp}(a/A)$ where $a \in V_i$ is not connected to any element in $\mathrm{acl}(A)$

(II) $\mathrm{tp}(a/A)$ where $a \in V_i$ is incident with some $b \in \mathrm{acl}(A) \cap V_j$ but not connected in $R(b)$ to any vertex in $\mathrm{acl}(A) \cap R(b)$.

(III) $\mathrm{tp}(a/A)$ where $a \in V_i$ is incident with some $x, y \in \mathrm{acl}(A)$ such that (x, a, y) is a flag with $x \in V_k, y \in V_j$; and as a special case of this we have

(IV) $\mathrm{tp}(a/A)$ where $a \in V_i$ has neighbors $x, y \in \mathrm{acl}(A)$ such that (x, a, y) is a (necessarily dense) flag.

By quantifier elimination any of these descriptions determines a complete type. Using the description of forking in Corollary 2.36 one sees easily that each of these types is regular and trivial.

Clearly, any type in (IV) has U-rank 1 and in fact Morley rank 1 by quantifier elimination. It also follows easily that $\mathrm{MR}(a/A) = \omega^n$ if $\mathrm{tp}(a/A)$ is as in (I). In case (II) we find that $\mathrm{MR}(a/A) = \omega^{n-j-1}$ or $\mathrm{MR}(a/A) = \omega^{j-1}$ depending on whether or not $i < j$. In case (III) we have $\mathrm{MR}(a/A) = \omega^{|k-j|-2}$.

Just as in [1] we obtain:

LEMMA 5.2. *Any regular type in T_n is nonorthogonal to a type as in (I), (II), or (III).*

PROOF. Let $p = \mathrm{tp}(b/\mathrm{acl}(B))$. If b is not connected to $\mathrm{acl}(B)$, then p is as in (I), so we may assume that $\mathrm{proj}(b/B) = C \neq \emptyset$. Let a be a vertex on a short path from b to C incident with an element of C. Then by Corollary 2.36 we see that p is nonorthogonal to $\mathrm{tp}(a/C)$ and $\mathrm{tp}(a/C)$ is of type (II) or (III). $\dashv$

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