MODEL THEORY I — EXERCISE 3

Question 1

Suppose $\varphi(x_0, \ldots, x_{n-1})$ is a formula and T is a theory. Prove that (1) and (2) are equivalent and do (3).

Hint: add new constant to the language.

(1) If $N, M \models T$, $N \subseteq M$ and $a_0, \ldots, a_{n-1} \in |N|$ then if $M \models \varphi(a_0, \ldots, a_{n-1})$ then $N \models \varphi(a_0, \ldots, a_{n-1})$.
(2) For some universal formula $\psi(x_0, \ldots, x_{n-1})$ (i.e., a formula of the form $\forall \bar{y} \theta(\bar{x}, \bar{y})$ where θ is quantifier free), $T \models \forall \bar{x}(\varphi \leftrightarrow \psi)$.
(3) Formulate and prove the analogous statement for existential formulas (i.e., formulas of the form $\exists \bar{y} \theta(\bar{x}, \bar{y})$ where θ is quantifier free).

Question 2

Complete the details of the following propositions.

(1) Here, $\mathrm{Diag}^{qf}(M)$ is what in Exercise 1 we called $\mathrm{Diag}(M)$. The following are equivalent for a theory T.
  (a) T is model complete: if $M \subseteq N$ are models of T then $M \prec N$.
  (b) For any model M of T, $T \cup \mathrm{Diag}^{qf}(M)$ is a complete theory in $L(|M|)$.
(2) The following are equivalent for a theory T.

  Hint: Try, instead of repeating the proof I gave in class, to start with (a) is equivalent to (d) and (b) implies (d) using Question 1.
  (a) T is model complete.
  (b) If $M_1 \subseteq M_2$ are models of T then M_1 is existentially closed in M_2: for every $a_0, \ldots, a_{n-1} \in |M_1|$ and every existential formula $\varphi(x_0, \ldots, x_{n-1})$, if $M_2 \models \varphi(a_0, \ldots, a_{n-1})$ then $M_1 \models \varphi(a_0, \ldots, a_{n-1})$.
  (c) Every formula $\varphi(\bar{x})$ is equivalent modulo T to an existential formula (i.e., there is some existential formula $\psi(\bar{x})$ such that $T \models \forall \bar{x}(\varphi \leftrightarrow \psi)$.
  (d) Same as (c) with universal instead of existential.

(e) $Mod(T)$ is the class of models of $T_\forall$ which are existentially closed in models of $T_\forall$ (M is said to be existentially closed in a class of models C if whenever $M \subseteq N$ and $N \in C$ then M is existentially closed in C).

Question 3 *

A sentence φ is called *equational* if it has the form $\forall \bar{x} \theta(\bar{x})$ where θ is a positive (i.e., no negation signs) quantifier free formula. Show that (1) and (2) are equivalent:

(1) φ is equational.
(2) Letting $C = Mod(\varphi)$, C is closed under substructures, images of homomorphisms and taking (finite or infinite) products. (This means, if $M \models \varphi$ and $h : M \to N$ is a surjective (onto) homomorphism, $N \models \varphi$, if $N \subseteq M \models \varphi$ then $N \models \varphi$ and if $M_i \models \varphi$ for $i \in I$ then $\prod M_i \models \varphi$.)

Possible hint: Enough to show that if M is a model of the theory of all equational sentences implied by φ then $M \models \varphi$.

Use a theorem of Birkhoff. Let C be a class of L-structures, closed under products, homomorphisms and substructures. Let X be an index set (finite or infinite). Construct the *free model of C on X* as follows:

- Find a sequence $\langle M_i \mid i \in I \rangle$ in C, such that any element of C generated by $\leq |X|$ elements is isomorphic to some M_i.
- Let $J = \{(i, f) \mid f : X \to M_i\}$. Let $M_{(i,f)} = M_i$. Let $M = \prod_{j \in J} M_j$. Define $F : X \to M$ by $F(x)(i, f) = f(x)$.
- Let N be the substructure of M generated by the image of F. Show that $N \in C$.
- Show that if $A \in C$ and $g : X \to A$ is any function, there exists a unique homomorphism $h : N \to A$ with $h \circ F = g$.

Question 4 (a little bit of set theory knowledge is required, namely transfinite induction arguments)

(1) Suppose that T is inductive and countable. Show that any model M of T can be embedded into a model of T which is existentially closed in models of T (call such a model *T-existentially closed*). Moreover, this model could be chosen to be of any cardinality $\kappa \geq \aleph_0$. (Side note: being T-ec is equivalent to being $T_\forall$-ec, see Question 2 (2.e))

Hint: find a model $M_1 \models T$ such that $M = M_0 \subseteq M_1$ and for any existential formula φ over M (i.e. in $L(|M|)$), if there is some $M_1 \subseteq N \models T$ such that $N \models \varphi$ then $M_1 \models \varphi$. Consider the countable union of models constructed that way.

(2) * Show Lindström's theorem: every inductive κ-categorical theory is model complete.

Hint: look at Exercise 1, Question 5 (2).

Question 5

(1) Suppose T is such that if $M \models T$ and $M \subseteq N$ then $N \models T$. Show that T is existential (i.e., for some T' consisting only of existential sentences, $T \equiv T'$).

Let $RCF = Th(\mathbb{R}, +, \cdot, 0, 1, <)$, and let RCF^- be its reduct to the field language (without the order).

(2) Show that RCF^- is model complete.

Hint: You are allowed to use Tarski's theorem which states that RCF eliminates quantifiers.

(3) * Show that RCF^- does not eliminate quantifiers. Namely, show that the formula $\exists y \left(y^2 = x\right)$ is not equivalent to a quantifier free formula.

Question 6

(1) Let T be a complete theory, and suppose $A, B \models T_\forall$. Show that there is $M \models T$ such that A, B are both substructures of M (by this I mean that both can be embedded into M).

(2) * Suppose T has QE (quantifier elimination).

(a) Show that $T_\forall$ has the amalgamation property (AP) where:

A theory T is said to have the *amalgamation property* if whenever $A, B, C \models T$ and $f : A \to B$ and $g : A \to C$ are embeddings, then there is a model $D \models T$ and embeddings $h_1 : B \to D$, $h_2 : C \to D$ such that $h_1 \circ f = h_2 \circ g$.

(b) If T has no constant symbols, then if T has QE then $T_\forall$ has the *joint embedding property* (JEP), which is exactly like AP, but we allow $A = \emptyset$.

Hint: Use (1).

(3) * Show that if T is model complete and such that $T_\forall$ has AP (and if there are no constants, also JEP), then for any structure $A \models T_\forall$, $T \cup \mathrm{Diag}^{qf}(A)$ is complete (and if there are no constants, T is complete).