MODEL THEORY I — EXERCISE 6

Question 1 (repeat question from Exercise 4)

Let $L = \{R\}$ where R is a binary relation symbol.

Let Gr be the theory of graphs (saying that R is symmetric, anti-reflexive).

Fix m, n and consider $\alpha_{m,n}$ which states:

$$\forall x_1 \dots x_m \forall y_1 \dots y_m \left(\bigwedge_{i,j} x_i \neq y_j \to \exists z \bigwedge_i R(x_i, z) \wedge \bigwedge_j \neg R(y_j, z) \wedge y_j \neq z \right).$$

(1) Let $N \in \mathbb{N}$ and let p_N be the probability that $\alpha_{m,n}$ holds in a graph G with vertex set $V_N = \{1, \dots, N\}$ (there are $2^{\binom{N}{2}}$ such graphs, and each is given the same probability). Show that $\lim_{n\to\infty} p_N = 1$.

Hint: fix $a_1, \dots, a_m, b_1, \dots, b_n \in V_N$ such that $a_{i'} \neq a_i \neq b_j \neq b_{j'}$ and compute the probability $p_N(\bar{a}, \bar{b})$ that $G \models \neg\exists z \bigwedge_i R(a_i, z) \wedge \bigwedge_j \neg R(b_j, z) \wedge b_j \neq z$. Show that it is $= c^{-(N-m-n)}$ where $c = 1 - 2^{-m-n} < 1$ (so exponentially goes to 0). Note that $1 - p_N \leq \sum_{\text{choices of } \bar{a},\bar{b}} p_N(\bar{a}, \bar{b})$. (You should be a bit careful as $\alpha_{m,n}$ didn't ask that $x_i \neq x_{i'}$.)

(2) Let $RG = \{\alpha_{m,n} \mid m, n \in \mathbb{N}\} \cup Gr$ (RG stands for Random Graph). Conclude that RG is consistent.

(3) Show by induction on $k < \omega$ that any finite graph of size k can be embedded in a model of RG. Conclude that $RG_\forall = Gr$.

(4) Show that RG eliminates quantifiers.

Question 2*

Let $T = Th(\mathbb{N}, +, \cdot, 0, 1, \leq)$ (or, if you prefer, Peano arithmetic). A model N is said to be an *end extension* of M iff $M \neq N$, $M \prec N$ and, for all $b \in N$ and $a \in A$, if $b < a$ then $b \in M$. Prove that every countable model of T has an end extension.

Hint: the omitting types theorem.

Question 3

(1) Let T be countable and consistent. Then any meager subset $X \subseteq S_n(T)$ can be omitted (see Exercise 5, Question 4).

(2) Find an example of an uncountable theory where the omitting types theorem fails.

Hint: look at the formulation of the omitting types theorem, there is a very simple example.

(3) Show that any complete theory whose language consists of only unary predicates has elimination of quantifiers.

Question 4

(1) Let Σ be a (consistent) theory in a countable language and let $S(\Sigma)$ be the set of its completions. Show that $S(\Sigma)$ is either countable or has size $2^{\aleph_0}$.

Hint: try to find a sentence α such that both $\Sigma \cup \{\alpha\}$ and $\Sigma \cup \{\neg\alpha\}$ have uncountably many completions.

(2) * Let Σ be as above. Let p be a partial type and let $S(\Sigma, p)$ be the set $\{Th(M) \mid M \models \Sigma, M \text{ omits } p\}$. Show that $S(\Sigma, p)$ is either countable or has size $2^{\aleph_0}$.

Question 5

Suppose G is a group acting on a set X (recall the definition of group action, if needed). We say that the action is *oligomorphic* if for all n the number of orbits of G acting on X^n is finite.

When M is a structure, we can think of $G = \operatorname{Aut}(M)$ as acting on M.

Show that the following are equivalent for a complete countable theory T.

(1) T is ω-categorical.
(2) Every countable model of T has an oligomorphic automorphism group.
(3) Some countable model of T has an oligomorphic automorphism group.

Question 6

(1) Show that if $L \subseteq L'$ and $T \subseteq T'$ are countable complete theories, and T' is ω-categorical, then so is T.
(2) Show that if T countable and is ω-categorical, $M \models T$ and $A \subseteq |M|$ is finite then $T_A = Th(M_A)$ is also ω-categorical.
(3) Does (2) hold with A infinite?

Question 7

Suppose T is a theory in L, not necessarily countable. We say that T is *small* if $S_n(T)$ is countable for all $n < \omega$.

(1) Suppose that T is small. Show that T is essentially countable: there is a countable sub-language $L_0 \subseteq L$ such that any formula $\varphi(\bar{x}) \in L$ is equivalent (modulo T) to some formula $\psi(\bar{x}) \in L_0$.
(2) Conclude that if $M \models T \upharpoonright L_0$ (so M is an L_0-structure) then M can be enriched to be a model of T.
(3) Show that if T is countable and not small then it has uncountably many countable models up-to isomorphism.

Question 8

Suppose T is a complete countable theory. Show that any countable model $M \models T$ has a countable elementary extension which is ω-homogeneous.

Question 9

(1) Let M be countable with a countable language and ($Th(M)$ is) ω-categorical. Show that if $X \subseteq M^n$ is invariant under the action of $\mathrm{Aut}(M)$ then X is definable (without parameters).
(2) * Show that if M is an infinite field, then $Th(M)$ is never ω-categorical.
(3) Suppose now that M is a structure (perhaps not ω-categorical). Assume that for some n only finitely many n-types are realized in M. Then in any $N \equiv M$, the exact same n-types are realized.