MODEL THEORY I — EXERCISE 2

Question 1

Suppose U is an ultrafilter on ω and that $\{Y_i \mid i < \omega\}$ is a family of sets of size $\leq \aleph_0$. Let $\mathcal{Y} = \prod_{i \in \omega} Y_i / D$.

(1) Is it possible that $\mathcal{Y}$ is finite?

(2) * Assuming that $\mathcal{Y}$ is infinite, show that $\mathcal{Y}$ is either countable or of size $2^{\aleph_0}$.

Hint: you may assume that $Y_i \subseteq \omega$ for all i, and even that Y_i is some ordinal (natural number in this case). Use the fact that there is a bijection (i.e., an injective surjective map) $c : \omega^{<\omega} \to \omega$ to find an injective function $j : \prod_{i \in \omega} Y_i \to \mathcal{Y}$.

If you can't do the general case, try to do it assuming that $Y_i = \aleph_0$ for all $i < \omega$. Also, see Question 5 (1).

Question 2

A theory T is called *pseudo finite* if for every sentence φ, if $T \models \varphi$ then there is a finite structure $M_0 \models \varphi$. Show that T is pseudo finite iff it has a model which is an ultraproduct of finite structures.

Question 3

Let X be a topological space, Y_1 and Y_2 are compact subspaces (perhaps not Hausdorff). Let $\mathcal{H}$ be a set of clopen (closed-open) subsets of X. Show that (1) and (2) are equivalent and answer (3).

(1) There is a positive Boolean combination (i.e., without taking complement) B of elements from $\mathcal{H}$ such that $Y_1 \subseteq B$ and $Y_2 \cap B = \emptyset$.

(2) For all $y_1 \in Y_1$ and $y_2 \in Y_2$ there is some $H \in \mathcal{H}$ such that $y_1 \in H$ and $y_2 \notin H$.

(3) * Show that this equivalence generalizes the separation lemma from class.

Question 4

(1) Is the result of Exercise 1, Question 0 (2) still true with just assuming that $M_i \subseteq M_{i+1}$ and $M_i \equiv M_{i+1}$?

(2) A formula is AE, or $\forall\exists$ if it is of the form $\forall x \exists y \varphi(x, y, z)$ where φ is quantifier free and x, y, z are <u>tuples</u> of variables (so really it can be $\forall x_1 \forall x_2 \exists y_1 \exists y_2 \varphi(x_1, x_2, y_1, y_2, z)$).

Suppose that T is a theory axiomatized by AE-sentences. Show that if $M_i \models T$ and $M_i \subseteq M_{i+1}$ as above, then $\bigcup_{i<\omega} M_i \models T$. (Theories that satisfy this property are called *inductive*.)

(3) Suppose ϕ is an existential formula (i.e., of the form $\exists x \varphi(x, y)$ where x and y are tuples of variables as in (2)), and suppose that ψ is a quantifier free formula. Write down an AE sentence that asserts that ϕ and ψ are equivalent.

Question 5

Do (1) + (2) or (3).

Two functions $f, g : \omega \to \omega$ are *almost disjoint* if $f(n) \neq g(n)$ for all but finitely many n's.

(1) Show that there are $2^{\aleph_0}$ almost disjoint functions.
(2) Let $\mathcal{F}$ be the set of all functions $\omega \to \omega$. Consider the structure $\mathcal{N}$ in the language $L = \{<\} \cup \{F_f \mid f \in \mathcal{F}\}$ where F_f is an unary function symbol, with universe ω, the usual order $<$ and such that $F_f^{\mathcal{N}} = f$. (To use a shorter notation, we can also describe $\mathcal{N}$ as $(\omega, <, \langle f \mid f \in \mathcal{F} \rangle)$.) Show that any proper elementary extension of $\mathcal{N}$ is uncountable.
(3) * Consider the structure M with universe $\mathbb{Q}$, the ordered field structure, and predicates P_r, Q_r for all $r \in \mathbb{R}$ where $P_r^M = \{q \in \mathbb{Q} \mid q < r\}$ and $Q_r^M = \{q \in \mathbb{Q} \mid q \geq r\}$. Show that any proper elementary extension of M is uncountable.

Question 6

(1) Is it possible that $(\mathbb{Z}^n, +) \cong (\mathbb{Z}^m, +)$ for $m \neq n$?
(2) What if you replace $\cong$ with $\equiv$ (elementary equivalence)?
(3) * What happens when you replace $\mathbb{Z}$ with $\mathbb{Q}$?