MODEL THEORY I — EXERCISE 4

Question 1

Let T be the theory of torsion free abelian groups (i.e., abelian groups in which $nx \neq 0$ for all $n, x \neq 0$) in the language $L = \{+, 0\}$.

Let T^* be the theory of divisible torsion free nontrivial abelian groups (i.e., such that for all x and n there is some y such that $ny = x$).

Prove that T^* is the model companion (in fact, model completion) of T.

Note: in class we already proved that T^* eliminates quantifiers, but we omitted some details which you should fill. You may write that proof in all detail, or write another proof.

Question 2

(1) Let $L = \{<\}$. Let LO be the theory of a linear order. Show that DLO is the model companion of T.
(2) Show that in fact DLO is the model completion of LO (i.e., DLO eliminates quantifiers).
(3) * Let $L' = \{<, f\}$ where f is a unary function. Let $T = LO \cup \{\forall xy\, (x < y \rightarrow f(x) < f(y))\}$ (i.e., f is order preserving). Show that T does not have a model companion.

   Hint: you should show that the class of existentially closed models of T (among models of T) is not an elementary class, i.e., for no theory $T \subseteq T^*$, is it the case that $Mod(T^*)$ is exactly the class of existentially closed models of T.

   Second hint: note that if $a < f(a)$ then for any $a < b < f(a)$, $b < f(b)$.

Question 3

A field K is called *algebraically closed* if for every polynomial $f(X) \in K[X]$ in one variable there is a solution in K.

Let L be the language of fields $\{+, \cdot, 0, 1\}$.

(1) Show that there is a first order theory in L, ACF which is the theory of all algebraically closed fields.

(2) Show that ACF eliminates quantifiers and conclude that ACF is the model completion of the theory of integral domains (you may assume that every field can be extended to an algebraically closed one).

Hint 1: if you know a bit of field theory (Mivnim 2), you may try to do a similar proof to the one we had in class for divisible torsion free abelian groups.

Hint 2: another approach is more elementary. Use the criterion we had in class. A primitive existential formula over a field A is (equivalent to) a formula of the form $\exists x \bigwedge f_i(x) = 0 \wedge \bigwedge g_j(x) \neq 0$ where f_i, g_j are in $A[X]$. Use the fact that in general, for two polynomials $h_1, h_2 \in K[X]$ over any field K, h_1, h_2 have a common zero a iff $h_1(a) = gcd(h_1, h_2)(a) = 0$.

(3) Is ACF complete? What are its completions?

Question 4

Let $L = \{E\}$ where E is a binary relation. Let EQ be the theory that says that E is an equivalence relation.

(1) Show that EQ has a model completion: it is the theory T saying that each E-class is infinite and that there are infinitely many classes.
(2) Suppose $EQ \subseteq T$ is a theory extending EQ which has quantifier elimination. Show that T is complete and describe all possible such T's.

Hint: consider the formula $\varphi(x)$ which says that x has at least m elements in its class.

(3) Is it true that all complete model complete theories extending EQ eliminate quantifiers?
(4) * Is it true that all complete model-complete theories extending $EQ \cup T_\infty$ eliminate quantifiers? (T_∞ is the theory saying that the universe is infinite).

Question 5

Show that if T eliminates quantifiers then it can be axiomatized by sentences of the form $\forall x_1 \forall x_2 \dots \forall x_n \exists y \psi$ where ψ is quantifier free. ($x_1, \dots, x_n$ and y are single variables.)

Question 6

Do (1) and (2) or (3).

For $0 \neq m \in \mathbb{N}$, let AB_m be the theory of abelian groups such that $\forall x (m \cdot x = 0)$.

(1) Show that when m is prime, AB_m has a model completion: $AB_m \cup T_\infty$ (see Question 4, (4)).
(2) Show that AB_6 has a model completion AB_6^*. (Hint: show that any group $G \models AB_6$ is isomorphic to $G_1 \times G_2$ where $G_1 \models AB_2$, $G_2 \models AB_3$.)
(3) * Show that AB_4 has a model completion.

Question 7

Let $L = \{R\}$ where R is a binary relation symbol.

Let Gr be the theory of graphs (saying that R is symmetric, anti-reflexive).

Fix m, n and consider $\alpha_{m,n}$ which states:

$$\forall x_1 \ldots x_m \forall y_1 \ldots y_m \left(\bigwedge_{i,j} x_i \neq y_j \rightarrow \exists z \bigwedge_i R(x_i, z) \wedge \bigwedge_j \neg R(y_j, z) \wedge y_j \neq z \right).$$

(1) Let $N \in \mathbb{N}$ and let p_N be the probability that $\alpha_{m,n}$ holds in a graph G with vertex set $V_N = \{1, \ldots, N\}$ (there are $2^{\binom{N}{2}}$ such graphs, and each is given the same probability). Show that $\lim_{n \to \infty} p_N = 1$.

Hint: fix $a_1, \ldots, a_m, b_1, \ldots, b_n \in V_N$ such that $a_{i'} \neq a_i \neq b_j \neq b_{j'}$ and compute the probability $p_N(\bar{a}, \bar{b})$ that $G \models \neg \exists z \bigwedge_i R(a_i, z) \wedge \bigwedge_j \neg R(b_j, z) \wedge b_j \neq z$. Show that it is $= c^{-(N-m-n)}$ where $c = 1 - 2^{-m-n} < 1$ (so exponentially goes to 0). Note that $1 - p_N \leq \sum_{\text{choices of } \bar{a}, \bar{b}} p_N(\bar{a}, \bar{b})$. (You should be a bit careful as $\alpha_{m,n}$ didn't ask that $x_i \neq x_{i'}$.)

(2) Let $RG = \{\alpha_{m,n} \mid m, n \in \mathbb{N}\} \cup Gr$ (RG stands for Random Graph). Conclude that RG is consistent.
(3) Show by induction on $k < \omega$ that any finite graph of size k can be embedded in a model of RG. Conclude that $RG_\forall = Gr$.
(4) Show that RG eliminates quantifiers.