MODEL THEORY I — EXERCISE 1

Question 0 (no need to submit, except for (3), (4) but if you want my feedback, you can)

1. (1) Complete the proofs of Tarski-Vaught test and the Downward and Upward Lowenheim-Skolem Theorems.
2. (2) Show that if $M_0 \prec M_1 \prec \ldots$ then their union $\bigcup M_i$ is an elementary extension of each M_i.
3. (3) Show that if M is a finite structure in a finite language L and $N \equiv M$ then $N \cong M$.
4. (4) * Generalize (3) for an infinite language.

Question 1

Suppose M, N are two L-structures. A map $f : A \to B$ where $A \subseteq M, B \subseteq N$ are two substructures (or empty) is called an *isomorphism of order* 0 if it is an isomorphism. It is an *isomorphism of order* $n+1$ if for any $a \in M$ there is an isomorphism $f' \supseteq f$ of order n such that $a \in \mathrm{Dom}\,(f')$ (i.e., in the domain of f'), and similarly for any $b \in N$ there is some $f' \supseteq f$ with the dual condition.

We say that f is an *isomorphism of order* ω if it is an isomorphism of degree n for all n.

We also allow $A = B = f = \emptyset$.

1. (1) Show that if the empty map from M to N is an isomorphism of order ω then $M \equiv N$ (M is elementarily equivalent to N, i.e., $Th\,(M) = Th\,(N)$).
2. (2) * Show that if the language $L = \{R\}$ where R is a binary relation, then the converse is also true, and moreover, if A, B are finite and $f : A \to B$ is an elementary map (i.e., a partial elementary map from M to N) then f is an isomorphism of order ω.

Question 2

1. (1) * Suppose C is a class of L-structures. Let $Th\,(C) = \{\varphi \mid \forall M \in C, M \models \varphi\}$. Show that M is a model of $Th\,(C)$ iff M is elementarily equivalent to an ultraproduct of elements of C.
2. (2) Show that C is an elementary class (see Question 3 below) iff C is closed under ultra products and elementary equivalence.

(3) Assume C is a class of finite structures containing only finitely many structures of size n for each $n < \omega$. Then the infinite models of $Th(C)$ are exactly the models of $\{\varphi \mid |\{M \mid M \models \neg\varphi\}| < \aleph_0\}$ (this is the set of all sentences for which almost all (= all but finitely many) structures in C are models.

Question 3

A class C of L-structures is an *elementary class* if it is the class of models of some (consistent) theory. It is called *finitely axiomatizable* if this theory can be chosen to be finite.

Show that C is finitely axiomatizable iff both C and its complement (i.e., all the L-structures not in C) are elementary classes.

Question 4 *

Let $L = \{0, <, f\}$ where 0 is a constant symbol, $<$ is a binary relation and f is an unary function symbol. Let M be an L structure with universe $\mathbb{R}$, with the usual interpretations of $<$ and 0 and such that $f^M(0) = 0$.

An element x in some $N \succ M$ is called an infinitesimal if $-1/n < x < 1/n$. Show that f^M is continuous in 0 iff for any elementary extension $N \succ M$, the map f^N takes infinitesimals to infinitesimals.

Question 5

Write an axiomatization for DLO (= Dense Linear Order) which is the theory of a dense linear order without endpoints.

Cantor's theorem states that if $M, N \models DLO$ are countable and $f : A \to B$, $A \subseteq M, B \subseteq N$ are finite sets, and f is order preserving, then f can be extended to an isomorphism from M to N.

(1) Conclude that DLO is complete.
(2) * Show that $M \subseteq N$ are any two models of DLO then $M \prec N$. (Hint: first reduce to the case where N is countable. Show that in general, if whenever $M \subseteq N$ are countable models of a countable[1] theory T then the same is true for any two such models: add a new unary predicates P to the theory and consider the theory of the structure N' with universe $|N|$ and $P^{N'} = M$.)

Question 6

[1]Abusing notation, the cardinality of a theory is defined as the number of formulas in its language, i.e., $|L| + \aleph_0$.

Show that $(\mathbb{Q},+)$ has no proper elementary substructure.

Question 7

Recall that if M is an L'-structure for $L' \supseteq L$, then we denote by $M \restriction L$ its L-reduct (which we get by forgetting the interpretations of all the symbols in $L' \backslash L$).

Given an L-structure M, and a subset $A \subseteq |M|$ (the universe of M), let $L(A) = L \cup \{c_a \mid a \in A\}$ where the c_a's are new constant symbols. Let M_A be the structure M enriched with interpretations of the constant symbols in the natural way: $c_a^{M_A} = a$.

(1) Let $Diag(M)$ be the set of atomic $L(|M|)$ sentences and their negations which hold in $M_{|M|}$. Show that N is an L-reduct of a model of $Diag(M)$ iff there is an embedding $h : M \to N$.
(2) Show that N is an L-reduct of a model of $Th\left(M_{|M|}\right)$ iff there is an elementary embedding $f : M \to N$.