# MODEL THEORY I — EXERCISE 14

There are five questions here. You may choose to do questions that you haven't solved from previous exercises instead of the ones here to get to five questions, as in Exercise 11.

Question 1

Suppose $T$ is totally transcendental.

(1) Prove with full details that every subset $A$ of $\mathfrak{C}$ has a constructible prime extension (we sketched this in class).
(2) Show that prime extensions are atomic.
(3) Suppose that $A$ has both a prime and a minimal extension (see Exercise 8, Question 4, i.e., a model $N$ containing $A$ such that if $M \prec N$ and $A \subseteq |M|$ then $M = N$). Show that then they are isomorphic over $A$ (i.e., there is an isomorphism fixing $A$ pointwise).
(4) (Without assuming that $T$ is totally transcendental) Show that the following are equivalent for a theory $T$ with monster $\mathfrak{C}$.
   (a) The isolated types are dense in $S_1(A)$ for all (small) sets $A \subseteq |\mathfrak{C}|$.
   (b) The isolated types are dense in $S_n(A)$ for all (small) sets $A \subseteq |\mathfrak{C}|$.

Question 2

For every countable theory $T$ the following are equivalent:

(1) Every set $A \subseteq |\mathfrak{C}|$ has a prime extension.
(2) Over every countable set $A \subseteq |\mathfrak{C}|$, the isolated types are dense.
(3) Over every set $A \subseteq |\mathfrak{C}|$, the isolated types are dense.

Hint: use all the knowledge we have on countable theories and their prime models. Also, look at the proof we had that totally transcendental theories have prime extensions over every set.

Question 3

Do (1) or (2).

(1) * Suppose that $M$ is a model and $\mathcal{I} \subseteq |M|$ is an indiscernible sequence $\langle a_i \mid i \in I \rangle$ where $I$ is some linear order. Let $P$ be a new predicate, and let $M_P$ be the structure expanding $M$ to $L \cup \{P\}$ such that $P^{M_P} = \mathcal{I}$. Let $N_P \succ M_P$. Show that some linear

order $<$ on $P^{N_P}$ which extends the linear order on $I$ and makes it into an indiscernible sequence in $N$.

(2) Suppose that $M$ and $\mathcal{I}$ are as above. Show that if $\mathcal{I}$ is indiscernible over $A$ then it is also indiscernible over $\operatorname{acl}(A)$ (recall the definition of acl from Exercise 13).

Hint: Work in $\mathfrak{C}$. Enlarge $\mathcal{I}$ so that it will have length $\kappa$ for some big enough $\kappa$. Take a model containing $A$. Then there must be two increasing tuples from $\mathcal{I}$ which will have the same type over $M$.

Question 4

(1) Let $L = \{E_n \mid n < \omega\}$ and let $T_2$ be a theory saying that $E_n$ is an equivalence relation such that $E_0$ has exactly two classes, $E_{n+1} \subseteq E_n$ and $E_{n+1}$ has two classes in each $E_n$-class.
(2) Show that $T_2$ is complete and has quantifier elimination.
(3) Show that $T_2$ is not $\omega$-stable, but stable at all cardinals $\kappa$ such that $\kappa \geq 2^{\aleph_0}$.
(4) * Now let $T_\infty$ say that $E_n$ is an equivalence relation such that $E_0$ has infinitely many classes, $E_{n+1} \subseteq E_n$ and $E_{n+1}$ has infinitely many classes in each $E_n$-class. Show that $T_\infty$ is complete, has quantifier elimination and is and $\kappa$-stable iff $\kappa^{\aleph_0} = \kappa$.

Question 5

(1) Show that the theory of the random graph, RG, is not $\kappa$-stable for any $\kappa$.
(2) Show that DLO is not $\kappa$-stable for any $\kappa$. Hint: You may use (3).
(3) * Show that for any cardinal $\mu$, there is a linear order of size $> \mu$ with a dense subset of size $\mu$.

Hint: let $\lambda$ be a minimal cardinal such that $\mu^{<\lambda} = \mu$ (why does it exist?).. Define a linear ordering on $\mu^{<\lambda}$ (the set of functions from some ordinal $\alpha < \lambda$ to $\mu$) in such a way that if $\mu^{<\lambda}$ is dense in $\mu^{<\lambda} \cup \mu^{\lambda}$.