MODEL THEORY I — EXERCISE 13

Question 1

Recall the definition of $\mathrm{dcl}_M(A)$ for a model M and a subset $A \subseteq M$. In class we talked of a monster model of a theory, $\mathfrak{C}$, which is a big saturated model. Once we fixed a theory, all sets and models considered will be small (i.e., of cardinality $< ||\mathfrak{C}||$) and contained in $\mathfrak{C}$.

We showed that if $M \prec N$ then $\mathrm{dcl}_M(A) = \mathrm{dcl}_N(A)$ so that we may define $\mathrm{dcl}(A) = \mathrm{dcl}_{\mathfrak{C}}(A)$ once we fixed the theory. We also gave an implicit argument that showed that $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$.

(1) Prove that equation explicitly.
(2) Prove that $c \in \mathrm{dcl}(A)$ iff there is an $\emptyset$-definable function $f : \mathfrak{C}^n \to \mathfrak{C}$ and some $\bar{a} \in A^n$ such that $c = f(\bar{a})$. (the number n can be also 0.)
(3) Work in $T = ACF_0$ (algebraically closed fields of char. 0). Show that for any set A, $\mathrm{dcl}(A)$ is the field generated by A.

    Hint: Use either the quantifier elimination in T to find a direct proof, or use Galois theory.
(4) * Now work in ACF_p for $p > 0$. Show that $\mathrm{dcl}(A) = \left\{c \,\middle|\, \exists n \in \mathbb{N} \left(c^{p^n} \in K\right)\right\}$ where K is the field generated by A.

Question 2

Recall Question 7 from Exercise 12.

Suppose that $M \prec \mathfrak{C}$ is a small model (where $\mathfrak{C}$ is as in Question 1). A partial type $p(x)$ over $\mathfrak{C}$ is *finitely satisfiable over M* if for every formula $\varphi(x, c) \in p$, for some $m \in M$ $\mathfrak{C} \models \varphi(m, c)$.

(1) Show that every type over M is finitely satisfiable over M.
(2) Suppose that p is a partial finitely satisfiable type over M, closed under finite conjunctions. Show that p can be extended to a complete type over $\mathfrak{C}$ which is still finitely satisfiable over M.

Hint: there are several ways to do this. One of them is to define F to be the collection of subsets of $M^{|x|}$ defined by p and noticing this is a filter.

(3) Show that any complete type $p \in S(\mathfrak{C})$ finitely satisfiable over M is invariant over M.

(4) * Let $p \in S(\mathfrak{C})$ be f.s. over M and let $\langle a_i \mid i < \omega \rangle$ be a *Morley sequence generated by* p as in Question 7, Exercise 12. Show that $\mathrm{dcl}(Ma_0 \dots a_{n-1}) \cap \mathrm{dcl}(Ma_n \dots a_{2n-1}) = M$.

Question 3

Suppose that M is any infinite structure. Show that M has a proper elementary extension N and an elementary embedding $f : N \to N$ such that $M = \bigcap_{n<\omega} f^{(n)}(N) = \{a \in N \mid f(a) = a\}$.

Hint: Use Question 3, (4) (note that there is a type $p \in S(\mathfrak{C})$, f.s. over M, containing $x \neq a$ for all $a \in M$), and Skolem hulls, See Exercise 12.

Question 4 *

Recall the definition of when a theory has built in definable Skolem functions — in Exercise 12, Question 2.

Show that if T has built in definable Skolem functions iff for all small sets A (see Question 1), $\mathrm{dcl}(A) \prec \mathfrak{C}$.

Question 5

(1) Suppose that T is a complete theory in L which is κ-stable for some κ. Show that there is some $L' \subseteq L$ of size $\leq \kappa$ such that for every formula $\varphi(\bar{x})$ in L, there is some formula $\varphi'(\bar{x}) \in L'$ such that $T \models \varphi \leftrightarrow \varphi'$ (i.e., T is a definable expansion of T').

(2) Show that T is totally transcendental (t. t.) iff $T \restriction L'$ is ω-stable for any countable $L' \subseteq L$.

(3) Conclude from (2) that if T is not t. t. then this is witnessed by formulas with one free variables.

(4) * Does (3) hold also for having a binary tree? Namely, suppose that T is countable and not small. Does it follow that there is a sequence of consistent formulas $\langle \varphi_s(x) \mid s \in 2^{<\omega} \rangle$ in one variable x over $\emptyset$ with $T \models \forall x \neg (\varphi_{s0}(x) \wedge \varphi_{s1}(x))$, $T \models \varphi_{s0} \vee \varphi_{s1} \to \varphi_s$.

Question 6

Let T be a complete theory with infinite models, and let $\mathfrak{C}$ be its monster model. For a small set A and a model M containing A, let $\mathrm{acl}_M(A)$ (the *algebraic closure of* A) be the set of elements $c \in M$ such that for some formula $\varphi(x) \in L_A$ and some $n < \omega$, $M \models \exists^{\leq n} x \varphi(x)$ and $M \models \varphi(c)$.

(1) Show that if $A \subseteq M \prec N$ then $\operatorname{acl}_M(A) = \operatorname{acl}_N(A)$. This allows us to denote $\operatorname{acl}(A) = \operatorname{acl}_{\mathfrak{C}}(A)$.

(2) Show that $c \in \operatorname{acl}(A)$ iff c has finitely many conjugates under the action of $\operatorname{Aut}(\mathfrak{C}/A)$, i.e., the set $\{\sigma(c) \mid \sigma \in \operatorname{Aut}(\mathfrak{C}/A)\}$ is finite.

(3) Show that $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$.

(4) Show that $\operatorname{acl}(A) = \bigcup \{\operatorname{acl}(F) \mid F \subseteq A \text{ finite}\}$.

(5) * Is it true that a $\mathfrak{C}$-definable set $D \subseteq |\mathfrak{C}|$ is definable over $\operatorname{acl}(A)$ iff it has finitely many conjugates over A (i.e., $\{\sigma(D) \mid \sigma \in \operatorname{Aut}(\mathfrak{C}/A)\}$ is finite)? (we proved an analogous result for dcl).

Hint: consider the theory of an equivalence relation with two infinite classes.

Question 7

This question has the same context as Question 6.

(1) Show that $\operatorname{acl}(A) = \bigcap \{M \mid A \prec M \prec \mathfrak{C}\}$.

Hint: use Question 6 (2).

(2) Show that in ACF_p for $p = 0$ or $p > 0$, model-theoretic algebraic closure is the same as the field-theoretic one: for a field $F \subseteq \mathfrak{C}$, $\operatorname{acl}(F)$ is F^a where F^a is the algebraic closure of F.

(3) * Let L contain the language of fields $\{0, 1, +, \cdot\}$, and let T be any L-complete theory of fields (with maybe more structure). Let $\mathfrak{C}$ be a monster for T. Show that for a field $F \subseteq \mathfrak{C}$, if $\operatorname{dcl}(F) = F$ then $\operatorname{acl}(F)$ is the field-theoretic algebraic closure of F inside $\mathfrak{C}$: the set of elements in $\mathfrak{C}$ that solve some polynomial over F.