MODEL THEORY I — EXERCISE 7

Question 1

Let $L = \{P, Q, f, g\}$ where P, Q are unary predicates, f, g are unary functions. Let T say that P and Q form a partition of the universe (i.e., disjoint and cover the universe), that $f : P \to P$ is injective with no cycles: for all $1 \leq n < \omega$, $T \models \forall x \left(f^{(n)}(x) \neq x\right)$. Also, $g : P \to Q$.

Remark: when I write $g : P \to Q$, you may take this to mean that g is the identity on Q and similarly for $f : P \to P$.

Show that T has a complete model companion T^*.

Hint: add a unary function symbol h and ask that $h \circ f = \mathrm{id}$. Write down axioms for the model *completion* in the new language (with h) and show that it eliminates quantifiers (so in particular also complete).

Question 2

In Question 3(2) of Exercise 6, you found an example of a theory which is not necessarily complete in an uncountable language, such that for some partial type Σ, there is no model omitting Σ even though Σ is not isolated. In this question you will strengthen this result.

(1) Show that for any uncountable κ there is a complete theory T of cardinality κ and a partial type Σ such that Σ is not isolated yet is realized in any model of T.

Hint: look at the theory from Question 1, and add κ many constants to Q and axioms saying that they are distinct. Choose Σ to be a type in two variables.

(2) * Improve this result (by maybe taking another complete theory) so that the type which cannot be omitted is also complete. (You don't have to prove all details if they are similar to the ones in Question 1).

Question 3*

Suppose T is a complete countable theory. Let D be a collection of types in finitely many variables in T, i.e., $D \subseteq \bigcup_n S_n(T)$. Suppose that M, N are models of T of cardinality $\kappa > |T|$ which are κ-homogeneous (if $f : A \to B$ is elementary where $A, B \subseteq |M|$ and $|A| < \kappa$, then for every $a \in M$ there is some $f' \supseteq f$ such that $a \in \mathrm{Dom}(f')$). Suppose also that for both models, the set of types in finitely many variables realized is exactly D.

Prove that $M \cong N$.

Hint: note that when $D = \bigcup_n S_n(T)$ then this just follows immediately from theorems we had in class.

You may find the notions of D-sets ($A \subseteq |M|$ for some $M \models T$ is a *D-set* if for every finite tuple from A, its type is in D), D-models (a model which is a D-set) and D-types (a type $p \in S_1(A)$ is a *D-type* if there is a realization $a \models p$ such that $A \cup \{a\}$ is a D-set) useful.

Question 4

Let T' be an L' theory and $T = T' \restriction L$. (Both are closed under logical implications.)

(1) Suppose that N' is a saturated model of T'. Show that $N = N' \restriction L$ is a saturated model of T.
(2) Now suppose that N is a saturated model of T. Show that if $||N|| = \lambda^+ = 2^\lambda$ and $|T'| < \lambda$ then N can be enriched to be a model of T'. Explain why you're allowed to assume that T, T' are complete.
(3) Do (2) without assuming the continuum hypothesis on λ, but just assuming that $|T'| \leq \lambda = ||N||$ is a regular cardinal.

    Hint: construct a continuous increasing sequence of models $M_i \models T'$ such that $||M_i|| \leq \lambda$ and M_i realizes all types in $S_1(A)$ for $|A| < \lambda$. In the successor step, embed $M_i \restriction L$ into N and show that $\mathrm{Diag}_L(N) \cup \mathrm{Diag}_{L'}(M_i)$ is consistent (as a theory in the appropriate language).
(4) In the next exercise you will prove (2) with no assumptions on λ (you can start thinking about it).

Question 5

Suppose M is saturated of cardinality κ.

(1) Suppose $A \subseteq |M|$ is such that $|A| < \kappa$. Then there is a type $p \in S_1(A)$ which has at least two realizations. (Try to improve it to get a type with κ realizations.)
(2) Show that $\mathrm{Aut}(M)$ has cardinality 2^κ.

Question 6

In (1) and (2), suppose that M is saturated of size λ.

(1) Show that M has a proper elementary extension N such that $M \cong N$.

(2) Show that any type in any number μ of variables over a set $A \subseteq M$ of size $< \lambda$ is realized, provided that $\mu \leq \lambda$.

(3) Let M be countable and ω-qf-homogeneous in a finite relational language (i.e., its language consists only of finitely many relation symbols), where ω-qf-homogeneous means: whenever $f : A \to B$ is a partial homomorphism (see Question 7(2)(b)), for every $a \in M$, there is some partial homomorphism containing f with a in its domain. Show that $Th(M)$ eliminates quantifiers and conclude that it is ω-categorical.

Question 7

Here you will show how working inside saturated structures sometimes makes life easier. For the following clauses assume that $M \models T$ is saturated of cardinality $|T| < \kappa$ and T is complete.

(1) Prove a version of the separation lemma for saturated models. Suppose that H is a set of formulas in $x_1, \ldots, x_n$ with no parameters containing $\bot, \top$ and closed under disjunctions and conjunctions. Suppose that Σ_1, Σ_2 are two sets of formulas in $x_1, \ldots, x_n$ with no parameters, closed under finite conjunctions. Then show that the following are equivalent:
   (a) For every tuple $\bar{a} = a_1, \ldots, a_n \models \Sigma_1$ and $\bar{b} = b_1, \ldots, b_n \models \Sigma_2$ from M, there is some formula from H which separates them, i.e., $M \models \varphi(\bar{a}) \wedge \neg\varphi(\bar{b})$ for some $\varphi \in H$.
   (b) There is some formula $\varphi(\bar{x}) \in H$ such that $\Sigma_1 \models \varphi$ and $\Sigma_2 \models \neg\varphi$ (this means that for every tuple from M satisfying Σ_1, it also satisfies φ, and similarly for Σ_2).
   (c) There are some formulas $\psi_1 \in \Sigma_1, \psi_2 \in \Sigma_2$ and $\varphi \in H$ such that $T \models \forall\bar{x}\,(\psi_1 \to \varphi) \wedge (\psi_2 \to \neg\varphi)$.

(2) Show that the following are equivalent.
   (a) T eliminates quantifiers.
   (b) For every finite $A, B \subseteq |M|$ and a map $f : A \to B$ which is a partial homomorphism from A onto B (for all $\bar{a}$ from A, and every relation symbol R, $R^M(\bar{a}) \Leftrightarrow R^M(\bar{b})$ and similarly for function symbols), there is an automorphism of M which expands f.

Question 8

Prove Beth Definability Theorem:

(1) Suppose T' is a complete L'-theory, $L \subseteq L'$, and $\varphi(\bar{x})$ is an L-formula. Then the following are equivalent:

   (a) Whenever $M_1, M_2 \models T'$ are such that $M_1 \restriction L = M_2 \restriction L$ (in particular they have the same universe), $\varphi^{M_1} = \varphi^{M_2}$.

   (b) For some L-formula ψ, $T' \models \forall \bar{x} \varphi \leftrightarrow \psi$.

   Instructions: you may assume that for some κ large enough, we have the continuum hypothesis. Use Question 7(1).

(2) * Prove (1) without the assumption that T' is complete.