MODEL THEORY I — EXERCISE 8

Question 1 (Presburger Arithmetic)

Complete all the missing details in the proof that $Th(\mathbb{Z}, +, -, 0, \langle P_n \mid 2 \le n < \omega \rangle)$ eliminates quantifiers, where P_n is a unary predicate interpreted as $P_n(m)$ iff n divides m and $'-'$ is a unary function symbol.

Write $x \equiv_n y$ for $P_n(x - y)$ (equality modulo n).

Use the following criterion (from Exercise 7, Question 7 (b)): suppose that M is a saturated model of some large cardinality, and $f : A \to B$ is a homomorphism, which you want to extend by adding some $a \in M$ to its domain.

Step 1: We may assume that A (and B) are subgroups such that if $n \cdot x \in A$ for some $x \in M$ then $x \in A$.

We may assume $a \notin A$. Let $q = \mathrm{tp}^{\mathrm{qf}}(a/A)$ be the quantifier-free type of a over A: it is (determined by) the set of formulas of the form $x = c$ and $mx \equiv_n c$ for $2 \le n < \omega, m \in \mathbb{Z}$ and $c \in A$ and their negations. We want to show that $f(q)$ is consistent.

Step 1.5: we can ignore formulas of the form $x \neq c$ as we can always find infinitely (hence $||M||$-many) solutions to the rest of $f(q)$, if one exists.

Step 2: use the Chinese Remainder Theorem to prove that it is enough to show that for every prime p, the set $r_p = f(q_p)$ is consistent, where q_p is the restriction of q to the language $L_p = \{+, -, 0, \langle P_{p^k} \mid k < \omega \rangle\}$. Here the saturation of M might help.

Step 3: Show that r_p is indeed consistent, by dividing into the following cases.

Case 1 — for some $c \in A$, $\neg P_p(c)$ (i.e., p does not divide c). Note that in this case, $M \models \forall x \exists! i < p^k \left(x \equiv_{p^k} ic \right)$. Let q'_p be the collection of formulas of the form $x \equiv_{p^k} ic$ for $i < p^k$ which belong to q_p. Then $q'_p \models q_p$ (i.e., all $d \in M$ which realizes q'_p, realizes q_p), and the same is true for $r'_p = f\left(q'_p\right)$. Explain all this, and show that r'_p is consistent.

Case 2 — every $d \in A$ is divisible by p, so the same is true in B. Let $q''_p \subseteq q_p$ be the set of formulas of the form $x \equiv_{p^k} 0$ and their negation. Then $q''_p \models q_p$ and the same is true for $r''_p = f\left(q''_p\right)$. Again show that r''_p is consistent.

Question 2

(1) Prove Robinsons's joint consistency lemma: suppose that L_1, L_2 are languages and $L = L_1 \cap L_2$. Suppose that T_1, T_2 are L_1, L_2 consistent theories such that $T = T_1 \cap T_2$ is complete (and consistent). Show that $T_1 \cup T_2$ is consistent.
(2) Show that the assumption that T is complete is necessary.
(3) Prove Craig's interpolation theorem: Suppose L_1, L_2 and L are as above. Suppose that $\varphi_i \in L_i$ for $i = 1, 2$ are sentences and that $\models \varphi_1 \to \varphi_2$ (in the language $L_1 \cup L_2$). Show that there is some sentence $\psi \in L$, $\models \varphi_1 \to \psi$ and $\models \psi \to \varphi_2$.

Hint: you may assume (for all questions) that for some κ large enough, we have the continuum hypothesis.

Question 3*

This question is a continuation of Exercise 7, Question 4. Let T' be an L' theory and $T = T' \restriction L$. (Both closed under logical implications.) Suppose that M is a saturated model of T of cardinality $\kappa \geq |T'|$ show that it can be enriched to be a model of T'.

You may assume that $\kappa > |T'|$ but the solution for $|T'| = \kappa$ is essentially the same.

Hints: Explain why we may assume that both T and T' are complete.

Enumerate $M = \{a_i \mid i < \kappa\}$, and add new constants c_i to the language to be interpreted as a_i. For a set C of constants, let $L'(C)$ be $L' \cup C$. Construct an increasing continuous sequence of sets $C_\alpha \subseteq \{c_i \mid i < \kappa\} = C_\kappa$ and consistent $L'(C_\kappa)$-theories T_α for $\alpha < \kappa$ all containing $\mathrm{Diag}_L(M) \cup T'$ such that:

- $T_\alpha \restriction L'(C_\alpha)$ is complete and $T_\alpha = T_\alpha \restriction L'(C_\alpha) \cup \mathrm{Diag}_L(M)$; $|C_\alpha| < \kappa$; If $\varphi(x) \in L'(C_{\alpha 2})$ then for some $i < \kappa$, $T_{\alpha 2+1}$ contains $\exists x \varphi(x) \to \varphi(c_i)$; $c_\alpha \in C_{\alpha 2+2}$.

You may assume, as usual, that for some λ as large as you want, $\lambda^+ = 2^\lambda$. In the $\alpha 2 + 1$ stage, let $\mathfrak{C}$ be a saturated model of $T_{\alpha 2+1}$ of size λ, then it is also a saturated model of T. Work within $\mathfrak{C}$ and use its homogeneity to define a model of $T_{\alpha 2+2}$.

Question 4

A structure is called *minimal* if it has no proper elementary substructures.

(1) Suppose that T is countable and complete. Show that if T has a prime model, then any minimal model is prime.
(2) Find an example of a countable atomic model in a countable language that is not minimal.

(3) * Show that while $(\mathbb{Z}, +, 0, 1)$ is both minimal and atomic, $(\mathbb{Z}, +, 0)$ is minimal yet not atomic (use Question (1)).

(4) Conclude that $Th\,(\mathbb{Z}, +, 0)$ has no prime model.

Question 5

(1) For each $n < \omega$, find a countable theory T which has exactly n models up to isomorphism.

Hint: consider $(\mathbb{Q}, <, P_0, \ldots, P_{n-2}, c_0, c_1, \ldots)$ where P_i form a partition of $\mathbb{Q}$ into dense subsets and $c_0 < c_1 < c_2 < \ldots$ all in P_0.

(2) Find an (incomplete) countable theory with exactly two models up to isomorphism.

(3) Show that a countable complete theory T is ω-categorical iff it has a model which is both prime and saturated.

(4) Show that the Ryll–Nardzewski theorem on countable ω-categorical theories is false in an uncountable language. I.e., find a complete theory T which has exactly one countable model up to isomorphism, but has infinitely many formulas modulo T.

Question 6

Let U be a non-principal ultrafilter on ω and let M_n for $n < \omega$ be a sequence of L-structure. Show that the ultraproduct $M = \prod_{n<\omega} M_n/U$ is countably saturated which means: every countable consistent set of formulas $\Gamma\,(x)$ with parameters from M is realized in M (consistent = every finite subset of Γ is realized in M). (Note that this is stronger than saying that M is $\aleph_0$-saturated if L is countable, and not as strong as saying that M is $\aleph_1$-saturated).

Hint: first, for every parameter $[f] \in M$ appearing in Γ fix a representative f. Enumerate $\Gamma = \{\varphi_i\,(x) \mid i < \omega\}$ where $\varphi_{i+1} \to \varphi_i$. Given $n < \omega$, let $a_n \in |M_n|$ be such that $a_n \models \varphi_i$ for the largest possible $i \leq n$ (make this statement precise).

Question 7

Let $L = \{P_n \mid n < \omega\}$ where P_n is a unary predicate. For $n < \omega$, let $L_n = \{P_k \mid k < n\}$. Let T_n be the empty theory in T_n, and similarly T is the empty theory in L.

(1) Show that for all $n < \omega$, T_n has a model completion T_n^* and write down axioms for it.

(2) Show that T_n^* is increasing with n and that $T^* = \bigcup T_n^*$ is the model completion of T.

(3) Show that T_n^* is ω-categorical yet T^* is not, and moreover, T^* has no prime model, and moreover no (consistent) formula in T^* isolates a complete type.