MODEL THEORY I — EXERCISE 10

Question 1 (o-minimality and definable functions in RCF)

Let $M = (\mathbb{R}, +, \cdot, 0, 1, <)$, and let RCF be its theory. Recall Tarski's theorem that states that M enjoys quantifier elimination.

(1) Show that every definable subset of $\mathbb{R}$ in M (with parameters) is a finite union of points and open intervals. Ordered structures in which every definable (with parameters) subset is a finite union of intervals and points are called *o-minimal*.

Let $L \supseteq \{+, \cdot, 0, 1, <\}$ and let N be an L-structure which is an o-minimal expansion of M: $|N| = \mathbb{R}$, $N \restriction \{0, 1, +, \cdot, <\} = M$, and N is o-minimal.

(2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a definable function in N (maybe with parameters). Show that for any open interval $U \subseteq \mathbb{R}$ there is some point $x \in U$ such that f is continuous at x.

Hint: this is an exercise in topology/calculus. First explain why you may assume that f has infinite range. Construct a decreasing sequence of open intervals $V_n \subseteq U$ such that $\overline{V_{n+1}} \subseteq V_n$, and $f(V_{n+1})$ is contained in an interval of diameter $1/n$. Finally, let $x \in \bigcap_n V_n$.

(3) Suppose that $N' \equiv N$. Suppose that $f : N' \to N'$ is a definable map (maybe with parameters). Show that for any open interval $U \subseteq N'$ there is some point $x \in U$ such that f is continuous at x (in the order topology on N').

(4) It is a fact that every $N' \equiv N$ must also be o-minimal. Conclude from it + the previous clause that if $f : N' \to N'$ is definable (maybe with parameters) then there is some finite partition of $|N'|$ to open intervals I_i and points x_j such that $f \restriction I_i$ is continuous.

Question 2

(1) Prove a version of Cantor-Bernstein for ω-categorical theories. Namely, suppose that T is ω-categorical and countable and that $\varphi(x)$, $\psi(y)$ are formulas (x and y may be tuples) and f, g are definable functions (not necessarily function symbols) such that $T \models$"$f : \varphi \to \psi$ is injective" and $T \models$"$g : \psi \to \varphi$ is injective". Then there is a

definable map h such that $T \models$"$h : \varphi \to \psi$ is a bijection". Hint: you may use Exercise 6, Question 9(1).

(2) * Show that this is not true in general.

Hint: one possible way to show this is to show that there is no definable bijection $f : (0,1) \to [0,1]$ in $(\mathbb{R}, +, 0, 1, \cdot, <)$. To do this, use Question 1, (4).

Question 3

In both (1) and (2), you don't have to be completely formal, writing all the details, and you can (in fact encouraged to) use drawings.

Let $L = \{<, \wedge\}$ and let Tr be the the theory of trees. Its axioms say that for all a, $\{x \mid x < a\}$ is linearly ordered and that $x \leq a \wedge b$ iff $x \leq a$ and $x \leq b$ (this is the meet function).

(1) Show that the class K of finite models of Tr has AP, HP and JEP and deduce (from a theorem we showed in class) that Tr has an ω-categorical model completion Tr^*.

(2) * Let $M \models Tr^*$ be countable (see Question 2). Prove that there are six kinds of indiscernible sequences of singletons $\langle a_i \mid i < \omega \rangle$ in M (e.g., increasing, decreasing, constant, ...). Write exactly what they are and prove that these are all such sequences.

(3) How many kinds of indiscernible sequences of singletons $\langle a_i \mid i < \omega \rangle$ are there in $(\mathbb{Q}, <)$?

Question 4

We work in the language $L = \{+, 0, 1, <\}$, with the structure $M = (\mathbb{R}, +, 0, 1, <)$.

(1) Show that M has no non-constant indiscernible sequence.

If $\Sigma(x_0, \ldots, x_{n-1})$ is a set of formulas, a sequence $\langle a_i \mid i \in I \rangle$ of elements of M (where $(I, <)$ is linearly ordered) is said to be Σ-*indiscernible* if for every $i_0 < \ldots < i_{n-1}$ and $j_0 < \ldots < j_{n-1}$ from I, and all $\varphi \in \Sigma$, $M \models \varphi(a_{i_0}, \ldots, a_{i_{n-1}})$ iff $M \models \varphi(a_{j_0}, \ldots, a_{j_{n-1}})$.

Let $\varphi(x, y, z, w) = x + y < z + w$. Suppose that $\Sigma(x, y, z, w)$ contains φ, $\varphi(x + x, y, z, w)$, $\varphi(x + x, y + y, z, w)$, ... (4 formulas, 5 including φ), and invariant under substituting any variable by a numeral: 0, 1, 2, ... and permuting the variables. For instance $x + y + y < z + z + 1$ is there but not $x + x + x + 0 < z + w$.

Suppose that $\langle a_i \mid i < \omega \rangle$ is Σ-indiscernible in M.

(2) Show that a_i is converging.

(3) Show that moreover $|a_{i+1} - a_i| < 2^{-i}$.

Question 5

(1) Suppose that K is algebraically closed. A polynomial map $p : K^n \to K^n$ is just a sequence $(p_0, \ldots, p_{n-1})$ of polynomials $p_j \in K[X_0, \ldots, X_{n-1}]$. Suppose that $p : K^n \to K^n$ is a polynomial map which is injective. Show that p is surjective.

Hint: in fact this is true for any definable map. Use the axiomatization of ACF.

(2) Show that any indiscernible sequence $\langle a_i \mid i < \omega \rangle$ in any model of $Th(\mathbb{Q}, +)$ satisfies $\{a_i \mid i < \omega\} = \{0\}$ or is an independent set (independent in the vector-space sense).

(3) * Show that the other direction is also true: every infinite independent set $\{a_i \mid i < \omega\}$ is an indiscernible sequence $\langle a_i \mid i < \omega \rangle$.

Question 6

Let T be a complete theory in a finite relational language and let M be an infinite model. Prove that the following are equivalent:

(1) T has quantifier elimination.

(2) Any isomorphism between finite substructures of M is elementary.

(3) The domain of any isomorphism between finite substructures of M can be expanded to include one more element.

Question 7

The following questions explains why Frank Ramsey came up with the notion of indiscernibles. Let L be a finite relational language. Assume that Σ is a universal theory, consisting of universal sentences with at most m universal quantifiers.

(1) Show that Σ has an infinite model iff it has a model whose universe is $\{a_0, \ldots, a_{m-1}\}$ such that the finite sequence $\langle a_i \mid i < m \rangle$ is indiscernible.

(2) Describe an algorithm for, given such a finite theory Σ determining whether it has an infinite model.

(3) * Let $L = \{R\}$ where R is binary. It is a fact (due to Gödel) that for all $L' \supseteq L$ there is no algorithm such that, given a sentence φ in L', returns "yes" iff φ is logically true ($\emptyset \models \varphi$). Show that there is no algorithm for determining, given a sentence φ in L, whether it has an infinite model.