# MODEL THEORY I — EXERCISE 9

Question 1

(1) Prove that the theory $DOAG$ of infinite divisible ordered abelian groups[1] in the language $\{+, 0, <\}$ is complete and has quantifier elimination.

(2) Show that $|S_1(DOAG)| = 3$ (i.e., $DOAG$ has exactly 3 complete one-types over $\emptyset$) but $S_2(DOAG) = 2^{\aleph_0}$ .

Question 2

Suppose that $T$ is a universal theory in a finite relational language such that its class of finite structures $K$ has AP and JEP.

(1) Show that $K$ has a Fraïssé limit $M$ (i.e., $M$ is a $K$-saturated structure).

(2) Show that $T^* = Th(M)$ is the model completion of $T$ so that $T^*_\forall = T$ (in particular $T$ has a complete model completion).

(3) Show that $T^*$ can be axiomatized as follows. For every finite models $N_0 \subseteq M_0 \models T$ such that $||M_0|| = ||N_0|| + 1$, write an axiom saying that if $x_0, \ldots, x_{k-1}$ is isomorphic to $N_0$ then there is some $x_k$ such that $x_0, \ldots, x_k$ is isomorphic to $M_0$.

Question 3

What do you get in Question 2 when you apply it to:

(In each case show **briefly** that Question 2 is applicable.)

(1) $T_=$ = the theory of equality.

(2) $T_E$ = the theory of equivalence relations in the language $\{E\}$.

(3) $T_{P_1,\ldots,P_n}$ = the theory of $n$ unary predicates $\{P_1, \ldots, P_n\}$.

(4) $T_<$ = the theory of linear orders in $\{<\}$.

(5) $T_{<_1,<_2}$ = the theory of two linear orders in the language $\{<_1, <_2\}$.

(6) $T_{<,P}$ = $<$ is a linear order and $P$ is a unary predicate.

(7) Show that $T^*_{P_1,\ldots,P_n} \subseteq T^*_{P_1,\ldots,P_m}$ for all $n \leq m$, and that $T^*_{<_1,\ldots,<_n} \subseteq T^*_{<_1,\ldots,<_m}$.

[1] An abelian group is *divisible* if for every non-zero element $b$ and every positive integer $n$ there is an element $c$ such that $nc = b$. It is ordered if it carries a linear ordering relation $<$ such that $a < b$ implies $a + c < b + c$ for all $a, b, c$.

Question 4

(1) Show that the assumptions of Question 2 hold for $T = \mathit{Graphs}$ in the language $L = \{R\}$ where $R$ is a binary relation symbol.
(2) Show that the axiomatization $\alpha_{n,m}$ which we had for the random graph in Exercise 6, Question 1 is equivalent to the axiomatization given in Question 2(4) for $T^*$.
(3) Conclude from the theorems we had in class that the theory $RG$ of the random graph is complete and has quantifier elimination.

Question 5

Recall the definition of a metric space $(X, d)$.

Let $L = \{R_q \mid q \in \mathbb{Q}\}$ where $R_q$ is a binary relation. Let $K$ be the class of finite metric spaces with rational distances (i.e., where $d$ maps into $\mathbb{Q}$). We consider spaces in $K$ as $L$-structures, interpreting $R_q(x, y)$ as $d(x, y) = q$.

(1) Show that $K$ is countable (has countably many isomorphism types).
(2) Show that $K$ has AP, JEP and HP.
(3) Conclude that $K$ has a Fraïssé limit $U_0$ which is a metric space that contains all finite metric spaces with rational distances. This space is called the *rational Urysohn space*.

Question 6 *

See Question 5. Let $U$ be the completion of $U_0$. Show that one $U$ contains a copy of all finite (in fact all separable) metric spaces. $U$ is called the *Urysohn space*.

Hint: prove the following lemma. For any finite subset $x_1, \ldots, x_n \in U$ and positive real numbers $\alpha_i$ for $i = 1, \ldots, n$ such that $|\alpha_i - \alpha_j| \leq d(x_i, x_j) \leq \alpha_i + \alpha_j$ there is some $y \in U$ such that $d(y, x_i) = \alpha_i$.

For the proof, first find $z_1, \ldots, z_n \in U_0$ and $\beta_i \in \mathbb{Q}$ such that $d(x_i, z_i)$ and $|\alpha_i - \beta_i|$ are very small and the appropriate inequalities still hold.

Next, find $y_1 \in U_0$ such that $d(z_i, y_1) = \beta_i$ so that $|d(y_1, x_i) - \alpha_i| < \alpha/2$ where $\alpha = \min\{\alpha_i\}$. Now add $y_1$ to the $x_i$'s and $\alpha/2$ to the $\alpha_i$'s.

Question 7

Do (3) + ((1) or (2))

(1) Prove that elementary embeddings have the amalgamation property: if $f : M_0 \to M_1$ and $g : M_0 \to M_2$ are elementary embedding then there is some $M_3$ and elementary embeddings $h : M_1 \to M_3$ and $r : M_2 \to M_3$ such that $h \circ f = r \circ g$.

(2) Suppose that $K$ is a class of finitely generated $L$-structures. Let $M_1, M_2$ be $K$-saturated. Without citing the theorem we proved in class saying that they are isomorphic, show that if $M_1$ is finitely generated, then so is $M_2$ and that they are isomorphic in this case.

(3) In class we showed that if $T$ is small then it has no binary tree of consistent formulas. Show that if $T$ is countable, then the converse is also true. (Hint: look at the solution to Exercise 6, Question 4).

Question 8 *

In class we showed that if $T$ is countable and the isolated types are dense, then $T$ has an atomic model. Prove that the same is true for theories of size $\aleph_1$ (remark: it stops being true for $\aleph_2$).

Hints:

Start with an $\aleph_1$-saturated model $M \models T$. A set $A \subseteq M$ is *atomic* if the type of every finite tuple from it is isolated.

(1) Show that it is enough to prove that if $A$ is atomic and countable and $\varphi(x) \in L(A)$ is a formula such that $M \models \exists x \varphi(x)$, then for some $c \in N$, $A \cup \{c\}$ is atomic and $M \models \varphi(c)$.

(2) Find such a $c$ as a realization of a countable type over $A$. Note that if $\operatorname{tp}(\bar{a})$ is isolated, then also $\operatorname{tp}(\bar{a}')$ for any subtuple $\bar{a}'$ of $\bar{a}$.