

Logik II: Model theory

Martin Bays

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[†]Dies Skript ist auch auf Deutsch verfügbar.

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1 Overview

- A key goal of model theory is to understand the *definable sets* of a structure $\mathcal{M}$, the subsets of powers $\mathcal{M}^n$ defined by formulas.
- Gödel vs Tarski: Some structures have uncontrollably wild definable sets, becoming ever more complicated as we allow more quantifiers; for example, in $(\mathbb{N}; +, \cdot)$, already with one unbounded existential quantifier we can define arbitrary recursively enumerable sets.

However, it is a remarkable fact that many structures important to mathematics avoid these Gödelian phenomena and are “tame”: their definable sets are well-controlled (in particular, all definable by formulas with only a few quantifiers¹), they often have decidable theories, and sometimes the models of the theory being classifiable up to isomorphism.

Examples include:

- $(\mathbb{C}; +, \cdot)$, $(\mathbb{F}_p^{\text{alg}}; +, \cdot)$, $(\mathbb{R}; +, \cdot)$, $(\mathbb{Q}_p; +, \cdot)$;
- $(\mathbb{Q}; <)$;
- Vector spaces;
- $(\mathbb{N}; +)$, $(\mathbb{N}; +, <)$, $(\mathbb{N}; \cdot)$ (but not $(\mathbb{N}; \cdot, <)$!);
- $(\mathbb{R}; +, \cdot, x \mapsto e^x)$;
- $(\mathbb{C}; +, \cdot, x \mapsto x^\zeta)$ (for most $\zeta \in \mathbb{C}$);
- Compact Lie groups e.g. $(\text{SO}_3(\mathbb{R}); *)$;
- $(\mathbb{F}_{p^*}; +, \cdot)$ for an “infinite” (pseudofinite) prime p^* ;
- Compact complex manifolds (complex tori, Calabi-Yau manifolds, etc);
- Differential and difference equations (in a certain sense);

We will examine only a fraction of this richness in this course, but we will develop tools with wide applicability.

- We often study a structure by considering other models of its theory. In particular, “tameness” of the class of models often corresponds to “tameness” of the definable sets. We will examine in detail the following strong form of this correspondence.

¹We consider e.g. $\exists x, y.$ as one quantifier rather than two.

A theory T is **κ -categorical** if it has a unique model of cardinality κ . Let $\mathcal{M}$ be an infinite structure in a countable language.

- Ryll-Nardzewski: $\text{Th}(\mathcal{M})$ is $\aleph_0$ -categorical iff for each $n \in \omega$ there are only finitely many $\mathcal{L}$ -definable subsets of $\mathcal{M}^n$.
- Baldwin-Lachlan: for an uncountable cardinal κ , $\text{Th}(\mathcal{M})$ is κ -categorical iff $\mathcal{M}$ is prime and minimal over a strongly minimal set defined over the prime model.

(We'll define these terms later; a strongly minimal set is a particularly straightforward structure, and being prime and minimal over it implies that the definable sets of $\mathcal{M}$ are “constructed” from its definable sets in a certain sense.)

2 Preliminaries

We work in ZFC throughout the course. A set A is *countable* if $|A| \leq \aleph_0$.

2.1 Syntax and structure

- A (first-order) (1-sorted) **language** is a set $\mathcal{L}$ of relation symbols, function symbols, and constants.
- An **$\mathcal{L}$ -structure** $\mathcal{M} = (M; (R^{\mathcal{M}})_R, (f^{\mathcal{M}})_f, (c^{\mathcal{M}})_c)$ is a non-empty set M equipped with interpretations of the symbols of $\mathcal{L}$:
 - $R^{\mathcal{M}} \subseteq \mathcal{M}^n$ for $R \in \mathcal{L}$ an n -ary relation symbol;
 - $f^{\mathcal{M}} : \mathcal{M}^n \rightarrow \mathcal{M}$ for $f \in \mathcal{L}$ an n -ary function symbol;
 - $c^{\mathcal{M}} \in \mathcal{M}$ for $c \in \mathcal{L}$ a constant.

Often, we write $\mathcal{M}$ to refer to the underlying set M .

We sometimes consider constants as 0-ary functions.

- An **$\mathcal{L}$ -term** is a variable, a constant, or $f(t_1, \dots, t_n)$ where t_i are $\mathcal{L}$ -terms and f is an n -ary function symbol.
- An **atomic $\mathcal{L}$ -formula** is $t_1 \doteq t_2$ or $R(t_1, \dots, t_n)$ or $\top$, where t_i are $\mathcal{L}$ -terms and R is an n -ary relation symbol. Here $\top$ is the *always true* sentence; $\mathcal{M} \models \top$ for any structure $\mathcal{M}$.
- An **$\mathcal{L}$ -formula** is an atomic $\mathcal{L}$ -formula or $\neg\phi$ or $(\phi \wedge \phi')$ or $\exists x.\phi$ where ϕ, ϕ' are $\mathcal{L}$ -formulas and x is a variable.
- An **$\mathcal{L}$ -sentence** is an $\mathcal{L}$ -formula with no free variables. For $\mathcal{M}$ an $\mathcal{L}$ -structure and σ an $\mathcal{L}$ -sentence, $\mathcal{M} \models \sigma$ is defined recursively.

- Abbreviations:

$$\begin{aligned}
(\phi \vee \psi) &\mapsto \neg(\neg\phi \wedge \neg\psi) \\
(\phi \rightarrow \psi) &\mapsto (\neg\phi \vee \psi) \\
(\phi \leftrightarrow \psi) &\mapsto ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)) \\
\forall x. \phi &\mapsto \neg\exists x. \neg\phi \\
x \neq y &\mapsto \neg x \doteq y \\
\perp &\mapsto \neg\top
\end{aligned}$$

- We also define abbreviations for conjunctions and disjunctions of finite sets of formulas :

$$\begin{aligned}
\bigwedge \emptyset &:= \top \\
\bigwedge (\Phi \cup \{\phi\}) &:= \left(\bigwedge \Phi \wedge \phi \right) \\
\bigvee \emptyset &:= \perp \\
\bigvee (\Phi \cup \{\phi\}) &:= \left(\bigvee \Phi \vee \phi \right).
\end{aligned}$$

- If $\mathcal{L}' \supseteq \mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}'$ -structure, we write $\mathcal{M}|_{\mathcal{L}}$ for the corresponding $\mathcal{L}$ -structure, and call $\mathcal{M}|_{\mathcal{L}}$ the **reduct** of $\mathcal{M}$ to $\mathcal{L}$, and $\mathcal{M}$ an **expansion** of $\mathcal{M}|_{\mathcal{L}}$ to $\mathcal{L}'$.
- If $\mathcal{M}$ is an $\mathcal{L}$ -structure and $A \subseteq \mathcal{M}$, the **expansion by constants for A** is the expansion $\mathcal{M}_A$ of $\mathcal{M}$ to $\mathcal{L}(A) := \mathcal{L} \dot{\cup} A$ defined by $a^{\mathcal{M}_A} := a$.
- We write a tuple $(a_1, a_2, \dots, a_n)$ ($n \geq 0$) as $\bar{a}$, and we write $|\bar{a}|$ for its length, $|\bar{a}| = n$. For A a set, we define $A^{<\omega} := \bigcup_{n \in \omega} A^n$, the set of tuples from A .
- We write an $\mathcal{L}$ -formula ϕ as $\phi(\bar{x})$ if $\bar{x}$ is a tuple of distinct variables and the free variables of ϕ are among $x_1, \dots, x_{|\bar{x}|}$. Then if $\mathcal{M}$ is an $\mathcal{L}$ -structure and $\bar{a} \in \mathcal{M}^{|\bar{x}|}$, we write $\phi(\bar{a})$ for the $\mathcal{L}(\mathcal{M})$ -sentence obtained by substituting a_i for x_i (for $i = 1, \dots, |\bar{x}|$). Then $\mathcal{M} \models \phi(\bar{a})$ means $\mathcal{M}_{\mathcal{M}} \models \phi(\bar{a})$.

Then the set **defined by $\phi(\bar{x})$ in $\mathcal{M}$** is

$$\phi(\mathcal{M}) := \{\bar{a} \in \mathcal{M}^{|\bar{x}|} : \mathcal{M} \models \phi(\bar{a})\} \subseteq \mathcal{M}^{|\bar{x}|}.$$

(Technically, this depends on the choice of tuple $\bar{x}$ and not just on ϕ).

Similarly, if $\phi(\bar{x}, \bar{y})$ is an $\mathcal{L}$ -formula and $\bar{a} \in \mathcal{M}^{|\bar{x}|}$, we write $\phi(\bar{a}, \bar{y})$ for the $\mathcal{L}(\mathcal{M})$ -formula obtained by substituting a_i for x_i .

- A **partial isomorphism** of $\mathcal{L}$ -structures is a partial function $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ such that for any atomic $\mathcal{L}$ -formula $\phi(\bar{x})$ and any $\bar{a} \in \text{dom } \theta^{|\bar{x}|}$, $\mathcal{M} \models \phi(\bar{a})$ iff $\mathcal{N} \models \phi(\theta(\bar{a}))$. If this holds for *any* $\mathcal{L}$ -formula ϕ , we call θ a **partial elementary map**.
- An **embedding** is a *total* partial isomorphism $\theta : \mathcal{M} \hookrightarrow \mathcal{N}$.

- An **elementary embedding** is a *total* partial elementary map $\theta : \mathcal{M} \xrightarrow{\sim} \mathcal{N}$.
- An **isomorphism** is a *surjective* embedding $\theta : \mathcal{M} \xrightarrow{\cong} \mathcal{N}$.
- A **substructure** (resp. **elementary substructure**), of an $\mathcal{L}$ -structure $\mathcal{N}$ is an $\mathcal{L}$ -structure $\mathcal{M}$ on a subset of $\mathcal{N}$ such that the inclusion $\iota : \mathcal{M} \rightarrow \mathcal{N}$ is an embedding (resp. an elementary embedding).

Convention: If $\mathcal{L}$ has no constants, we also allow the “empty structure” $\emptyset$ as an $\mathcal{L}$ -substructure of any $\mathcal{L}$ -structure (even though $\emptyset$ is not an $\mathcal{L}$ -structure!²).

- We write $\mathcal{M} \leq \mathcal{N}$ for a substructure³, and $\mathcal{M} \preceq \mathcal{N}$ for an elementary substructure.
- If $A \subseteq \mathcal{M}$ is a subset of an $\mathcal{L}$ -structure $\mathcal{M}$, let

$$\langle A \rangle_{\mathcal{L}}^{\mathcal{M}} := \bigcap \{ \mathcal{B} : A \subseteq \mathcal{B} \leq \mathcal{M} \} \leq \mathcal{M}$$

be the $\mathcal{L}$ -substructure generated by A .

Note: when $\mathcal{L}$ has no constant symbols, $\langle \emptyset \rangle_{\mathcal{L}}^{\mathcal{M}} = \emptyset$.

Lemma 2.1. $|\langle A \rangle_{\mathcal{L}}^{\mathcal{M}}| \leq \max(|A|, |\mathcal{L}|, \aleph_0)$.

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Proof. Let $A_0 := A$ and, considering constants as 0-ary functions,

$$A_{i+1} := A_i \cup \{f^{\mathcal{M}}(\bar{a}) : f \in \mathcal{L} \text{ an } n\text{-ary function symbol, } \bar{a} \in A_i^n, n \geq 0\}.$$

Then $\langle A \rangle_{\mathcal{L}}^{\mathcal{M}} = \bigcup_{i \in \omega} A_i$, and $|A_{i+1}| \leq |A_i| + |\mathcal{L}| \cdot \max(|A|, \aleph_0) \leq \max(|A_i|, |\mathcal{L}|, \aleph_0)$. Hence $|A_i| \leq \max(|A|, |\mathcal{L}|, \aleph_0)$ for all i . So $|\bigcup_i A_i| \leq \max(|A|, |\mathcal{L}|, \aleph_0)$. $\square$

- We can always make an embedding into an inclusion by applying an isomorphism:

Lemma 2.2. *Suppose $\theta : \mathcal{A} \hookrightarrow \mathcal{B}$ is an embedding of $\mathcal{L}$ -structures. Then there is an isomorphism $\sigma : \mathcal{B} \rightarrow \mathcal{B}'$ such that $\mathcal{A} \leq \mathcal{B}'$ and $\sigma \circ \theta = \text{id}_{\mathcal{A}}$.*

Proof. First, let $\sigma : \mathcal{B} \rightarrow \mathcal{B}'$ be a bijection with a set $B' \supseteq A$ such that $\sigma \circ \theta = \text{id}_{\mathcal{A}}$. Let $\mathcal{B}'$ be the $\mathcal{L}$ -structure on B' such that σ is an isomorphism. Then $\text{id}_{\mathcal{A}} = \sigma \circ \theta$ an embedding, so $\mathcal{A} \leq \mathcal{B}'$. $\square$

²In many ways it would be preferable to allow the empty structure to be a structure, and some authors do this.

³Many authors write $\mathcal{M} \subseteq \mathcal{N}$ for the substructure relation.

2.2 Theories

- An **$\mathcal{L}$ -theory** is a set of $\mathcal{L}$ -sentences.
- The **theory** of an $\mathcal{L}$ -structure $\mathcal{M}$ is

$$\text{Th}(\mathcal{M}) := \{\sigma : \mathcal{M} \models \sigma, \sigma \text{ is an } \mathcal{L}\text{-sentence}\}.$$

- An $\mathcal{L}$ -structure $\mathcal{M}$ is a **model** of an $\mathcal{L}$ -theory T ,

$$\mathcal{M} \models T,$$

if $\mathcal{M} \models \sigma$ for all $\sigma \in T$.

- $T \models \sigma$ means: $\mathcal{M} \models \sigma$ for any $\mathcal{M} \models T$. We also write $T \vdash \sigma$.

$T \models T'$ or $T \vdash T'$ means: $T \models \sigma$ for all $\sigma \in T'$.

$T \models_{T''} T'$ or $T \vdash_{T''} T'$ means: $T \cup T'' \models T'$.

- T is **consistent** if it has a model.

Remark 2.3. T is consistent iff $T \not\vdash \perp$.

- A consistent $\mathcal{L}$ -theory T is **complete** if for any $\mathcal{L}$ -sentence σ

$$T \models \sigma \text{ or } T \models \neg\sigma.$$

- $\mathcal{L}$ -structures $\mathcal{M}, \mathcal{N}$ are **elementarily equivalent**, $\mathcal{M} \equiv \mathcal{N}$, if $\text{Th}(\mathcal{M}) = \text{Th}(\mathcal{N})$.

Remark 2.4. A consistent theory T is complete iff $\mathcal{M} \equiv \mathcal{N}$ whenever $\mathcal{M}, \mathcal{N} \models T$.

Remark 2.5. If A is a common subset of $\mathcal{L}$ -structures $\mathcal{M}$ and $\mathcal{N}$, then $\text{id}_A : \mathcal{M} \dashrightarrow \mathcal{N}$ is partial elementary iff $\mathcal{M}_A \equiv \mathcal{N}_A$.

In particular, if $\mathcal{M} \subseteq \mathcal{N}$, we have $\mathcal{M} \preceq \mathcal{N}$ iff $\mathcal{M}_{\mathcal{M}} \equiv \mathcal{N}_{\mathcal{M}}$.

The $\mathcal{L}_{\mathcal{M}}$ -theory $\text{Th}(\mathcal{M}_{\mathcal{M}})$ is called the **elementary diagram** of $\mathcal{M}$.

3 Ultraproducts

Let I be a set. A non-empty set $\mathcal{B} \subseteq \mathcal{P}(I)$ is a **filter base** if

- $X, Y \in \mathcal{B} \Rightarrow \exists Z \in \mathcal{B}. X \cap Y \supseteq Z$;
- $\emptyset \notin \mathcal{B}$.

A maximal filter base $\mathcal{U}$ is an **ultrafilter**; equivalently: $\mathcal{U}$ is a filter base, and for any $X \subseteq I$ gilt

$$X \in \mathcal{U} \text{ or } I \setminus X \in \mathcal{U}.$$

As an immediate consequence of Zorn's Lemma, we have

Fact 3.1. Any filter base $\mathcal{B} \subseteq \mathcal{P}(I)$ extends to an ultrafilter $\mathcal{B} \subseteq \mathcal{U} \subseteq \mathcal{P}(I)$.

Remark 3.2. Fact 3.1 is not a theorem of ZF, and it is strictly weaker than the axiom of choice modulo ZF.

Remark 3.3. Ultrafilters are *upwards-closed*: if $X \subseteq Y$ then $X \in \mathcal{U} \Rightarrow Y \in \mathcal{U}$. An upwards-closed filter base is a *filter*.

If $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter and a_i are elements of sets A_i ($i \in I$), the **ultralimit** $\lim_{i \rightarrow \mathcal{U}} a_i$ be the equivalence class $(a_i)_i / \sim^{\mathcal{U}}$ of the sequence $(a_i)_i$ under the equivalence relation

$$(a_i)_i \sim^{\mathcal{U}} (a'_i)_i \text{ iff } \{i : a_i = a'_i\} \in \mathcal{U}.$$

If A_i are sets, the **ultraproduct** is the set of all ultralimits,

$$\prod_{i \rightarrow \mathcal{U}} A_i := \Pi_{i \in I} A_i / \sim^{\mathcal{U}} = \left\{ \lim_{i \rightarrow \mathcal{U}} a_i : a_i \in A_i \right\}.$$

We have $\lim_{i \rightarrow \mathcal{U}} (a_i, b_i) = (\lim_{i \rightarrow \mathcal{U}} a_i, \lim_{i \rightarrow \mathcal{U}} b_i)$.

For functions $f_i : A_i \rightarrow B_i$ we define

$$\lim_{i \rightarrow \mathcal{U}} f_i : \prod_{i \rightarrow \mathcal{U}} A_i \rightarrow \prod_{i \rightarrow \mathcal{U}} B_i$$

by

$$(\lim_{i \rightarrow \mathcal{U}} f_i)(\lim_{i \rightarrow \mathcal{U}} a_i) := \lim_{i \rightarrow \mathcal{U}} f_i(a_i).$$

If $(M_i : i \in I)$ are $\mathcal{L}$ -structures, the **ultraproduct** $\mathcal{M} = \prod_{i \rightarrow \mathcal{U}} M_i$ is the $\mathcal{L}$ -structure such that

- as sets, $\mathcal{M} := \prod_{i \rightarrow \mathcal{U}} M_i$;
- $\mathcal{M} \models R(\lim_{i \rightarrow \mathcal{U}} \bar{a}_i) \Leftrightarrow \{i : M_i \models R(\bar{a}_i)\} \in \mathcal{U}$ (for $R \in \mathcal{L}$ a relation symbol);
- $f^{\mathcal{M}} := \lim_{i \rightarrow \mathcal{U}} f^{M_i}$ (for $f \in \mathcal{L}$ a function symbol);
- $c^{\mathcal{M}} := \lim_{i \rightarrow \mathcal{U}} c^{M_i}$ (for $c \in \mathcal{L}$ a constant symbol).

Theorem 3.4 (Łoś). *For $\mathcal{L}$ -structures M_i , an $\mathcal{L}$ -formula $\phi(\bar{x})$, and $\bar{a}_i \in M_i$,*

$$\prod_{i \rightarrow \mathcal{U}} M_i \models \phi(\lim_{i \rightarrow \mathcal{U}} \bar{a}_i) \text{ iff } \{i : M_i \models \phi(\bar{a}_i)\} \in \mathcal{U}.$$

Proof. Exercise. □

An ultrafilter $\mathcal{U}$ is **principal** if there is $i_0 \in I$ such that $\mathcal{U} = \{X \subseteq I : i_0 \in X\}$. Then $\prod_{i \rightarrow \mathcal{U}} M_i \cong M_{i_0}$.

Example 3.5. Let $\mathcal{U} \subseteq \mathcal{P}(P)$ be a non-principal ultrafilter on the set $P \subseteq \mathbb{N}$ of primes. Then the ultraproduct of finite fields $\prod_{p \rightarrow \mathcal{U}} (\mathbb{F}_p; +, \cdot)$ is a field of characteristic 0 (a “pseudofinite field”).

If $\mathcal{U} \subseteq \mathcal{P}(I)$ is an ultrafilter and $\mathcal{M}$ is an $\mathcal{L}$ -structure, the **ultrapower** is defined as $\mathcal{M}^{\mathcal{U}} := \prod_{i \rightarrow \mathcal{U}} \mathcal{M}$.

Lemma 3.6. *With respect to the diagonal embedding $a \mapsto \lim_{i \rightarrow \mathcal{U}} a$, $\mathcal{M}$ is an elementary substructure, $\mathcal{M} \preceq \mathcal{M}^{\mathcal{U}}$.*

Proof. Exercise. □

Example 3.7. Let $\mathcal{U} \subseteq \mathcal{P}(\omega)$ be a non-principal ultrafilter. Let $(\mathbb{R}^*; +, \cdot) := (\mathbb{R}; +, \cdot)^{\mathcal{U}}$ (a “non-standard real field”). Let

$$\epsilon := \lim_{n \rightarrow \mathcal{U}} \frac{1}{n+1} \in \mathbb{R}^*.$$

Then $0 < \epsilon < r$ for all $r \in \mathbb{R} \subseteq \mathbb{R}^*$.

4 Compactness

Theorem 4.1 (Compactness). *Suppose every finite subset of an $\mathcal{L}$ -theory T is consistent. Then T is consistent.*

Proof. Let $\mathcal{P}^{\text{fin}}(T) := \{T' \in \mathcal{P}(T) : |T'| < \aleph_0\}$ be the set of finite subsets of T . Say $\mathcal{M}_{T'} \models T'$ for $T' \in \mathcal{P}^{\text{fin}}(T)$.

For $T' \in \mathcal{P}^{\text{fin}}(T)$, let $[T'] = \{T'' \in \mathcal{P}^{\text{fin}}(T) : T' \subseteq T''\} \subseteq \mathcal{P}^{\text{fin}}(T)$. Let $\mathcal{B} = \{[T'] : T' \in \mathcal{P}^{\text{fin}}(T)\} \subseteq \mathcal{P}(\mathcal{P}^{\text{fin}}(T))$. Then $\mathcal{B}$ is a filter base, since $[T'] \cap [T''] = [T' \cup T'']$, and $[T'] \neq \emptyset$ since $T' \in [T']$, and $\mathcal{B} \neq \emptyset$ since $[\emptyset] \in \mathcal{B}$. By Fact 3.1, let $\mathcal{U} \supseteq \mathcal{B}$ be an ultrafilter on $\mathcal{P}^{\text{fin}}(T)$.

Let $\mathcal{M} = \prod_{T' \rightarrow \mathcal{U}} \mathcal{M}_{T'}$.

Then for $\sigma \in T$,

$$\{T' : \mathcal{M}_{T'} \models \sigma\} \supseteq \{T' : \sigma \in T'\} = [\{\sigma\}] \in \mathcal{U},$$

so by Łoś, $\mathcal{M} \models \sigma$.

So $\mathcal{M} \models T$, so T is consistent. $\square$

Lemma 4.2 (Separation). *Let T_1 and T_2 be consistent $\mathcal{L}$ -theories, and Σ a set of $\mathcal{L}$ -sentences closed under $\wedge$ and $\vee$. Then TFAE:*

- (i) *Given $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$, there exists $\sigma \in \Sigma$ with $\mathcal{M}_1 \models \sigma$ and $\mathcal{M}_2 \models \neg\sigma$.*
- (ii) *There exists $\sigma \in \Sigma$ with $T_1 \models \sigma$ and $T_2 \models \neg\sigma$.*

We then say Σ **separates** T_1 from T_2 .

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Proof.

(i) $\Leftarrow$ (ii): Clear.

(i) $\Rightarrow$ (ii): Let $\mathcal{M}_1 \models T_1$. For $\mathcal{M}_2 \models T_2$ there is by (i) $\sigma_{\mathcal{M}_2} \in \Sigma$ with $\mathcal{M}_1 \models \sigma_{\mathcal{M}_2}$ and $\mathcal{M}_2 \models \neg\sigma_{\mathcal{M}_2}$. Then $T_2 \cup \{\sigma_{\mathcal{M}_2} : \mathcal{M}_2 \models T_2\}$ is inconsistent, so by compactness there is a finite conjunction $\sigma_{\mathcal{M}_1}$ of the $\sigma_{\mathcal{M}_2}$, such that $T_2 \models \neg\sigma_{\mathcal{M}_1}$. Then $\sigma_{\mathcal{M}_1} \in \Sigma$ by closedness of Σ under $\wedge$, and $\mathcal{M}_1 \models \sigma_{\mathcal{M}_1}$.

Now $T_1 \cup \{\neg\sigma_{\mathcal{M}_1} : \mathcal{M}_1 \models T_1\}$ is inconsistent, so by compactness and closedness of Σ under $\vee$, there is a finite disjunction $\sigma \in \Sigma$ of the $\sigma_{\mathcal{M}_1}$, such that $T_1 \models \sigma$. Then $T_2 \models \neg\sigma$.

$\square$

5 Quantifier elimination

5.1 Definitions

Definition 5.1. A formula is **quantifier free** (**qf**) if it contains no quantifiers

Definition 5.2. $\mathcal{L}$ -formulas $\phi(\bar{x})$ and $\psi(\bar{x})$ are **equivalent modulo** an $\mathcal{L}$ -theory T , written $\phi(\bar{x}) \leftrightarrow_T \psi(\bar{x})$, if

$$T \models \forall \bar{x}. (\phi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Similarly, $\phi(\bar{x}) \rightarrow_T \psi(\bar{x})$ if $T \models \forall \bar{x}. (\phi(\bar{x}) \rightarrow \psi(\bar{x}))$.

(As always, we allow here the case $|\bar{x}| = 0$, i.e. the case that ϕ and ψ are sentences.)

Definition 5.3. An $\mathcal{L}$ -theory T has **quantifier elimination (QE)** if any $\mathcal{L}$ -formula $\phi(\bar{x})$ is equivalent modulo T to a quantifier free formula $\psi(\bar{x})$.

An $\mathcal{L}$ -structure $\mathcal{M}$ has QE if $\text{Th}(\mathcal{M})$ has QE.

5.2 Discussion

Remark 5.4. An $\mathcal{L}$ -structure $\mathcal{M}$ has QE iff every $\mathcal{L}$ -definable set is defined by a qf $\mathcal{L}$ -formula.

Example 5.5. $(\mathbb{R}; +, -, \cdot, 0, 1)$ does not have QE: The order $x < y$ is definable by $\exists z. (z \neq 0 \wedge x + z \cdot z = y)$ but not by any qf formula.

Remark 5.6. If $\mathcal{L}$ has no constants, then the only qf sentences up to equivalence are $\top$ and $\perp$. So any consistent $\mathcal{L}$ -theory with QE is complete.

Example 5.7. Let $\mathcal{L}_\emptyset$ be the empty language $\mathcal{L}_\emptyset := \emptyset$. We will see below that if X is an infinite set, the $\mathcal{L}_\emptyset$ -structure $(X;)$ has QE. Moreover, we will see that $T_\infty := \{\exists x_1, \dots, x_n. \bigwedge_{i \neq j} x_i \neq x_j : n \in \omega\}$ has QE, and so axiomatises $\text{Th}((X;))$.

If $\mathcal{M}$ is an $\mathcal{L}$ -structure and Φ is a set of $\mathcal{L}$ -formulas, the **expansion by relations for Φ** is the expansion of $\mathcal{M}$ to $\mathcal{L} \cup \{R_\phi : \phi \in \Phi\}$, where R_ϕ is interpreted as $\phi(\mathcal{M})$. It has the same definable sets as $\mathcal{M}$.

If $\mathcal{M}$ does not have QE, we can try to expand by relations for non-qf formulas until we obtain QE. e.g.:

Fact 5.8 (Tarski-Seidenberg). $(\mathbb{R}; +, -, \cdot, 0, 1, <)$ has QE.

We can always do this in a trivial way:

Remark 5.9. For any $\mathcal{L}$ -structure $\mathcal{M}$, the expansion by relations for *all* $\mathcal{L}$ -formulas (the “Morleyisation” of $\mathcal{M}$) has QE.

For some particularly intractable structures, we can’t really do any better than Morleyising:

Fact 5.10 (“The arithmetic hierarchy is strict.”). *For any $n \in \omega$, the expansion of $(\mathbb{N}; +, \cdot)$ by relations for all formulas with at most n unbounded quantifiers (equivalently, all formulas of the form $\exists \bar{x}_1. \neg \exists \bar{x}_2. \dots \neg \exists \bar{x}_n. \phi$, where ϕ has no unbounded quantifier) does not have QE.*

5.3 Criterion for QE

Definition 5.11. • A **basic formula**⁴ is an atomic formula or the negation of an atomic formula.

⁴Also known as a *literal*.

- A formula $\phi(\bar{x})$ is **primitive existential** if it is of the form $\exists y. \bigwedge_i \psi_i(y, \bar{x})$, where each ψ_i is basic.

Lemma 5.12. *Let T be an $\mathcal{L}$ -theory. If any primitive existential $\mathcal{L}$ -formula $\phi(\bar{x})$ is equivalent modulo T to a quantifier free formula $\psi(\bar{x})$, then T has QE.*

Proof. We show by induction on complexity that any $\phi(\bar{x})$ is equivalent modulo T to a quantifier free formula $\psi(\bar{x})$. For atomic ϕ this is clear. For $\phi = \phi' \wedge \phi''$ or $\phi = \neg\phi'$ it is immediate by induction.

For $\phi = \exists y. \phi'$: by the inductive hypothesis $\phi' \leftrightarrow_T \psi$ with ψ quantifier-free. We may assume ψ is in disjunctive normal form : $\psi = \bigvee_i \bigwedge_j \psi_{ij}$, where each ψ_{ij} is basic. So $\phi \leftrightarrow_T \bigvee_i \exists y. \bigwedge_j \psi_{ij}$. Each formula $\exists y. \bigwedge_j \psi_{ij}$ is primitive existential, and we conclude by the assumption. $\square$

Definition 5.13. The **diagram** of an $\mathcal{L}$ -substructure $\mathcal{A}$ of an $\mathcal{L}$ -structure is the $\mathcal{L}(\mathcal{A})$ -theory

$$\text{Diag}(\mathcal{A}) := \text{qfTh}(\mathcal{A}_{\mathcal{A}}) := \{\sigma : \sigma \text{ qf } \mathcal{L}(\mathcal{A})\text{-sentence, } \mathcal{A}_{\mathcal{A}} \models \sigma\}.$$

Lemma 5.14 (Method of diagrams). *Up to isomorphism, the models of $\text{Diag}(\mathcal{A})$ are precisely $\mathcal{M}_{\mathcal{A}}$, where $\mathcal{M}$ is an $\mathcal{L}$ -structure and $\mathcal{A} \leq \mathcal{M}$.*

Theorem 5.15. *For an $\mathcal{L}$ -theory T , TFAE:*

- (i) T has QE.
- (ii) If $\mathcal{M}, \mathcal{N} \models T$ have a common $\mathcal{L}$ -substructure $\mathcal{A}$, then $\mathcal{M}_{\mathcal{A}} \equiv \mathcal{N}_{\mathcal{A}}$.
- (ii') " T is **substructure complete**": If $\mathcal{M} \models T$ and $\mathcal{A} \leq \mathcal{M}$ is an $\mathcal{L}$ -substructure, then $T \cup \text{Diag}(\mathcal{A})$ is complete.
- (iii) If $\mathcal{M}, \mathcal{N} \models T$ have a common $\mathcal{L}$ -substructure $\mathcal{A}$, and $\bar{a} \in \mathcal{A}^{<\omega}$ and $\phi(\bar{x})$ is a primitive existential $\mathcal{L}$ -formula, then

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a}).$$

- (iii') If $\mathcal{M}, \mathcal{N} \models T$ have a common finitely generated $\mathcal{L}$ -substructure $\mathcal{A}$, and σ is a primitive existential $\mathcal{L}(\mathcal{A})$ -sentence, then

$$\mathcal{M}_{\mathcal{A}} \models \sigma \Leftrightarrow \mathcal{N}_{\mathcal{A}} \models \sigma.$$

Proof.

(i) $\Rightarrow$ (ii): Let $\phi(\bar{a})$ be an $\mathcal{L}(\mathcal{A})$ -sentence. By (i), $\phi(\bar{x})$ is equivalent to some qf $\phi'(\bar{x})$ modulo T . Then $\mathcal{M}_{\mathcal{A}} \models \phi'(\bar{a}) \Leftrightarrow \mathcal{A} \models \phi'(\bar{a}) \Leftrightarrow \mathcal{M}_{\mathcal{B}} \models \phi'(\bar{a})$. So $\mathcal{M}_{\mathcal{A}} \models \phi(\bar{a}) \Leftrightarrow \mathcal{M}_{\mathcal{B}} \models \phi(\bar{a})$.

(ii) $\Leftrightarrow$ (ii'): By Lemma 5.14, the models of $T \cup \text{Diag}(\mathcal{A})$ are exactly $\mathcal{M}_{\mathcal{A}}$, where $\mathcal{M} \models T$ and $\mathcal{A} \leq \mathcal{M}$. So (ii) exactly says that $T \cup \text{Diag}(\mathcal{A})$ is complete.

End of lecture 03

(ii) $\Rightarrow$ (iii): Clear.

(iii) $\Leftrightarrow$ (iii'): Clear.

(iii) $\Rightarrow$ (i): Let $\phi(\bar{x})$ be primitive existential. By Lemma 5.12, it suffices to show that ϕ is equivalent modulo T to a qf formula .

Let $\bar{c}$ be a tuple of new constants with $|\bar{c}| = |\bar{x}|$.

Let $T_1 := T \cup \{\phi(\bar{c})\}$ and $T_2 := T \cup \{\neg\phi(\bar{c})\}$.

If T_1 is inconsistent, then $T \models \forall \bar{x}. \neg\phi(\bar{x})$, and so $\phi(\bar{x}) \leftrightarrow_T \perp$.

If T_2 is inconsistent, then $T \models \forall \bar{x}. \phi(\bar{x})$, and so $\phi(\bar{x}) \leftrightarrow_T \top$.

So assume T_1 and T_2 are consistent.

Suppose $\Sigma := \{\psi(\bar{c}) : \psi(\bar{x}) \text{ qf } \mathcal{L}\text{-formula}\}$ does not separate T_1 from T_2 . Then by Lemma 4.2, there are $\mathcal{M}_1, \mathcal{M}_2 \models T$ and $\bar{a}_i \in \mathcal{M}_i$ with $\mathcal{M}_1 \models \phi(\bar{a}_1)$ and $\mathcal{M}_2 \models \neg\phi(\bar{a}_2)$, but for $\psi(\bar{x})$ qf, $\mathcal{M}_1 \models \psi(\bar{a}_1) \Leftrightarrow \mathcal{M}_2 \models \psi(\bar{a}_2)$.

But then the map $\bar{a}_1 \mapsto \bar{a}_2$ extends to an isomorphism $\langle \bar{a}_1 \rangle_{\mathcal{L}}^{\mathcal{M}_1} \xrightarrow{\cong} \langle \bar{a}_2 \rangle_{\mathcal{L}}^{\mathcal{M}_2}$ (namely $t^{\mathcal{M}_1}(\bar{a}_1) \mapsto t^{\mathcal{M}_2}(\bar{a}_2)$ for t an $\mathcal{L}$ -term), which itself extends to an isomorphism $\mathcal{M}_1 \xrightarrow{\cong} \mathcal{M}'_1 \geq \langle \bar{a}_2 \rangle_{\mathcal{L}}^{\mathcal{M}_2}$ (by Lemma 2.2). Then $\mathcal{M}'_1 \models \phi(\bar{a}_2)$ and $\mathcal{M}_2 \models \neg\phi(\bar{a}_2)$. But this contradicts (iii).

So there is $\psi(\bar{c}) \in \Sigma$ such that $T_1 \models \psi(\bar{c})$ and $T_2 \models \neg\psi(\bar{c})$, i.e. $\phi(\bar{x}) \rightarrow_T \psi(\bar{x})$ and $\neg\phi(\bar{x}) \rightarrow_T \neg\psi(\bar{x})$. So $\phi(\bar{x}) \leftrightarrow_T \psi(\bar{x})$.

□

We can now generalise Remark 5.6 to arbitrary languages:

Corollary 5.16. *Let T be a consistent $\mathcal{L}$ -theory with QE. Then TFAE:*

(i) T is complete.

(ii) For any $\mathcal{M}, \mathcal{N} \models T$, we have $\langle \emptyset \rangle_{\mathcal{L}}^{\mathcal{M}} \cong \langle \emptyset \rangle_{\mathcal{L}}^{\mathcal{N}}$.

Proof.

(i) $\Rightarrow$ (ii): The map $t^{\mathcal{M}} \mapsto t^{\mathcal{N}}$ for t an $\mathcal{L}$ -term with no variables is an $\mathcal{L}$ -isomorphism.

(ii) $\Rightarrow$ (i): Let $\mathcal{M} \models T$ and $\mathcal{A} := \langle \emptyset \rangle_{\mathcal{L}}^{\mathcal{M}} \leq \mathcal{M}$. Let $\mathcal{N} \models T$. By (ii) there is $\mathcal{N}' \cong \mathcal{N}$ with $\mathcal{A} \leq \mathcal{N}'$. Then by Theorem 5.15(ii) $\mathcal{N}_{\mathcal{A}} \equiv \mathcal{M}'_{\mathcal{A}}$. Hence $\mathcal{N} \equiv \mathcal{M}' \cong \mathcal{M}$. So T is complete.

□

5.4 Examples

5.4.1 T_{∞}

Sei T_{∞} die $\mathcal{L}_{\emptyset}$ -Theorie

$$T_{\infty} := \{\exists x_1, \dots, x_n. \bigwedge_{i \neq j} x_i \neq x_j : n \in \omega\}.$$

Proposition 5.17. T_{∞} ist vollständig und besitzt Quantorenelimination.

Proof. Vollständigkeit folgt aus Quantorenelimination, weil die Sprache keine Konstante hat.

Für QE zeigen wir Theorem 5.15(iii').

Seien $\mathcal{M}, \mathcal{N} \models T_{\infty}$ und $\mathcal{A} \leq \mathcal{M}, \mathcal{N}$ eine endliche gemeinsame Teilmenge.

Sei $\exists y. \psi(y)$ eine primitiv existentielle $\mathcal{L}_\infty(\mathcal{A})$ -Aussage. Angenommen, dass $b \in \mathcal{M}$ existiert, sodass $\mathcal{M}_\mathcal{A} \models \psi(b)$. Wir müssen ein $b' \in \mathcal{N}$ finden, sodass $\mathcal{N}_\mathcal{A} \models \psi(b')$.

Falls $b \in \mathcal{A}$: Setze $b' := b$. Falls $b \notin \mathcal{A}$: Weil $\mathcal{N} \models T_\infty$, ist $\mathcal{N}$ unendlich; setze $b' \in \mathcal{N} \setminus \mathcal{A}$.

Dann ist $\text{id}_\mathcal{A} \cup \{b \mapsto b'\}$ eine Bijektion und demnach ein $\mathcal{L}_\infty(\mathcal{A})$ -Isomorphismus. Daher $\mathcal{N}_\mathcal{A} \models \psi(b')$. $\square$

5.4.2 DLO

Let $\mathcal{L}_< := \{<\}$ and let DLO be the $\mathcal{L}_<$ -theory of dense linear orderings without endpoints :

$$\begin{aligned} \text{DLO} := \{ & \forall x, y, z. (\neg x < x \\ & \wedge (x < y \vee x = y \vee y < x) \\ & \wedge ((x < y \wedge y < z) \rightarrow x < z) \\ & \wedge (x < y \rightarrow \exists w. (x < w \wedge w < y)) \\ & \wedge \exists w. w < x \\ & \wedge \exists w. x < w) \}. \end{aligned}$$

Proposition 5.18. DLO ist vollständig und besitzt Quantorenelimination.

Insbesondere ist DLO eine Axiomatisierung von $(\mathbb{Q}; <)$. Folglich ist $\text{Th}((\mathbb{Q}; <))$ entscheidbar.

Proof. Vollständigkeit folgt aus Quantorenelimination, weil die Sprache keine Konstante besitzt.

Entscheidbarkeit folgt aus Vollständigkeit, weil DLO eine rekursive Menge ist, und damit $\text{Th}((\mathbb{Q}; <)) = \{\sigma : \text{DLO} \models \sigma\}$ und ihr Komplement $\{\sigma : \text{DLO} \not\models \sigma\}$ rekursiv aufzählbar sind, und daher $\text{Th}((\mathbb{Q}; <))$ rekursiv ist.

Seien $\mathcal{M}, \mathcal{N} \models \text{DLO}$ und $\mathcal{A} = \{a_1, \dots, a_n\} \leq \mathcal{M}, \mathcal{N}$ eine gemeinsame endliche Unterstruktur. Ohne Beschränkung gilt $a_1 < a_2 < \dots < a_n$.

Sei $\exists y. \psi(y)$ eine primitiv existentielle $\mathcal{L}_<(\mathcal{A})$ -Aussage. Angenommen, dass $b \in \mathcal{M}$ existiert, sodass $\mathcal{M}_\mathcal{A} \models \psi(b)$. Wir finden ein $b' \in \mathcal{N}$ mit $\mathcal{N}_\mathcal{A} \models \psi(b')$.

Es gibt vier Fälle:

- (i) $b \in \mathcal{A}$: Setze $b' = b$.
- (ii) $b < a_1$: Sei $b' \in \mathcal{N}$ sodass $b' < a_1$ (b' existiert, weil $\mathcal{N}$ keinen Endpunkt hat).
- (iii) $b > a_n$: Sei $b' \in \mathcal{N}$ sodass $b' > a_n$ (b' existiert, weil $\mathcal{N}$ keinen Endpunkt hat).
- (iv) $a_i < b < a_{i+1}$: Sei $b' \in \mathcal{N}$ sodass $a_i < b' < a_{i+1}$ (b' existiert, weil $\mathcal{N}$ dicht ist).

In allen Fällen ist $A \cup \{b\}$ zu $A \cup \{b'\}$ über A isomorph als angeordnete Mengen. Daher $\mathcal{N}_\mathcal{A} \models \psi(b')$. $\square$

5.4.3 $(\mathbb{Z}; S)$

Let $\mathcal{L}_S := \{S\}$, where S is a unary function.

Let T_S die $\mathcal{L}_S$ -Theorie von zyklischen Bijektionen

$$T_S := \{\forall x, y. ((S(x) = S(y) \rightarrow x = y) \wedge \exists z. S(z) = x)\} \cup \{\forall x. S^n(x) \neq x : n \geq 1\}$$

(where $S^{n+1}(x) := S(S^n(x))$; $S^1(x) := S(x)$).

$(\mathbb{Z}; S) \models T_S$ (where $S^{\mathbb{Z}}(n) := n + 1$).

Proposition 5.19. T_S ist vollständig und besitzt Quantorenelimination.

Insbesondere ist $(\mathbb{Z}; S)$ entscheidbar und durch T_S axiomatisiert.

Proof. Vollständigkeit folgt aus Quantorenelimination, weil die Sprache keine Konstante besitzt.

Seien $\mathcal{M}, \mathcal{N} \models T_S$ mit einer endlich erzeugten gemeinsamen Unterstruktur $\mathcal{A}$.

Wir können annehmen, dass $S(\mathcal{A}) = \mathcal{A}$: In der Tat ist $\bigcup_n (S^{\mathcal{M}})^{-n}(\mathcal{A})$ zu $\bigcup_n (S^{\mathcal{N}})^{-n}(\mathcal{A})$ über $\mathcal{A}$ $\mathcal{L}_S$ -isomorph, weil $\mathcal{M}$ und $\mathcal{N}$ zyklischfrei sind.

Dann ist jede atomare $\mathcal{L}_S(\mathcal{A})$ -Formel $\phi(x)$ modulo $T_S \cup \text{Diag}(\mathcal{A})$ zu $x \doteq a$ (für ein $a \in \mathcal{A}$) oder $\top$ oder $\perp$ äquivalent. In der Tat:

$$S^n(x) \doteq S^m(x) \leftrightarrow_{T_S} \begin{cases} \top & (n = m) \\ \perp & (n \neq m) \end{cases};$$

$$S^n(x) \doteq S^m(a) \leftrightarrow_{T_S \cup \text{Diag}(\mathcal{A})} x \doteq S^{m-n}(a) (\in \mathcal{A}).$$

Daher ist eine primitiv existentielle $\mathcal{L}_S(\mathcal{A})$ -Aussage σ zu $\top$ oder $\perp$ oder

$$\exists y. \bigwedge_{i < k} y \doteq a_i \wedge \bigwedge_{i < l} y \not\doteq b_i$$

modulo $T_S \cup \text{Diag}(\mathcal{A})$ äquivalent. Weil $\mathcal{M}$ und $\mathcal{N}$ unendlich sind, gilt $\mathcal{M}_{\mathcal{A}} \models \sigma \Leftrightarrow \mathcal{N}_{\mathcal{A}} \models \sigma$.

Die Behauptung folgt nach Theorem 5.15(iii'). $\square$

5.4.4 ACF

Sei $\mathcal{L}_{\text{ring}} := \{+, -, \cdot, 0, 1\}$. Sei ACF die $\mathcal{L}_{\text{ring}}$ -Theorie von algebraisch abgeschlossenen Körpern:

$$\text{ACF} := [\text{Körperaxiome}] \cup \{\forall z_0, \dots, z_n. \exists x. \sum_{i=0}^n z_i x^i \doteq 0 : n \geq 1\}.$$

Proposition 5.20. ACF besitzt Quantorenelimination.

Proof. Sei $K_i \models \text{ACF}$ und $R = \langle a_1, \dots, a_n \rangle \leq K_1, K_2$ ein endlich erzeugter gemeinsamer Unterring.

Sei $\exists y. \psi(y)$ eine primitiv existentielle $\mathcal{L}_{\text{ring}}(R)$ -Aussage. Angenommen, dass $b \in K_1$ existiert, sodass $K_1 \models \psi(b)$. Wir zeigen, dass $K_2 \models \exists y. \psi(y)$.

Sei F_i der Quotientenkörper in K_i von R . Dann lässt id_R sich zu einem Isomorphismus $f : F_1 \xrightarrow{\cong} F_2$ erweitern ($F_1 \ni \frac{r}{s} \mapsto \frac{r}{s} \in F_2$).

Nun sei G_i der algebraische Abschluss von F_i in K_i , d.h. die Menge aller Lösungen von Polynomgleichungen mit Koeffizienten in F_i .

Weil der algebraisch Abschluss von F_i bis auf F_i -Isomorphie eindeutig bestimmt ist, lässt f sich zu einem Isomorphismus $g : G_1 \xrightarrow{\cong} G_2$ erweitern.

Falls $b \in G_1$, gilt $K_2 \models \psi(g(b))$.

Sonst: b ist transzendent über G_1 . Dann ist $G_1(b)$ isomorph über G_1 zum rationalen Funktionkörper $G_1(X)$. Sei K'_2 eine echte elementare Erweiterung von K_2 . Dann gibt es $b' \in K'_2 \setminus G_2$. Dann ist $G_2(b')$ wieder isomorph über G_2 zum rationalen Funktionkörper $G_2(X)$. Deshalb lässt g sich zu einem Isomorphismus $h : G_1(b) \rightarrow G_2(b')$ mit $h(b) = b'$ erweitern. Deswegen folgt $K'_2 \models \psi(b')$ und deshalb $K'_2 \models \exists y. \psi(y)$. Daher folgt schließlich $K_2 \models \exists y. \psi(y)$. $\square$

Für $p \in \mathbb{N}$ prim, sei $\text{ACF}_p := \text{ACF} \cup \{\bar{p} \doteq 0\}$, wobei $\bar{n}$ der Term $1 + 1 + \dots + 1$ (n Mal) ist.

Sei $\text{ACF}_0 := \text{ACF} \cup \{\bar{n} \neq 0 : n \geq 1\}$.

Theorem 5.21. *Die Vervollständigungen von ACF sind ACF_p für p prim oder 0.*

Proof. Die Charakteristik eines Körpers ist prim oder 0. Für K ein Körper mit Charakteristik p ist

$$\langle \emptyset \rangle_{\mathcal{L}_{\text{ring}}}^K = \begin{cases} \mathbb{F}_p & p \neq 0 \\ \mathbb{Z} & p = 0 \end{cases}.$$

Deswegen impliziert Quantorenelimination nach Corollary 5.16 Vollständigkeit von jedem ACF_p . $\square$

Theorem 5.22 (Ax). *Any injective polynomial map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ (i.e. $F(\bar{a}) = (F_1(\bar{a}), \dots, F_n(\bar{a}))$, where $F_i \in \mathbb{C}[X]$) is surjective.*

Proof.

Claim 5.23. *Let p be prime. Any injective polynomial map $F : (\mathbb{F}_p^{\text{alg}})^n \rightarrow (\mathbb{F}_p^{\text{alg}})^n$ is surjective.*

Proof. Recall: $\mathbb{F}_p^{\text{alg}} = \bigcup_k \mathbb{F}_{p^k}$.

Let k_0 be such that the coefficients of F are in $\mathbb{F}_{p^{k_0}}$.

Let $k \geq k_0$. Dann $F(\mathbb{F}_{p^k}^n) \subseteq \mathbb{F}_{p^k}^n$, and so by injectivity and the pigeonhole principle, $F(\mathbb{F}_{p^k}^n) = \mathbb{F}_{p^k}^n$.

Hence $F((\mathbb{F}_p^{\text{alg}})^n) = (\mathbb{F}_p^{\text{alg}})^n$. $\square$

Let $n, d \in \omega$. Let $\sigma_{n,d}$ be an $\mathcal{L}_{\text{ring}}$ -sentence expressing that any injective polynomial map $F : K^n \rightarrow K^n$ consisting of polynomials of degree $\leq d$ is surjective:

$$\sigma_{n,d} := \forall z_{1,0}, \dots, z_{n,d}. (\forall \bar{x}, \bar{y}. ((\bigwedge_i \sum_j z_{i,j} x_i^j \doteq \sum_j z_{i,j} y_i^j) \rightarrow \bigwedge_i x_i \doteq y_i) \rightarrow \forall \bar{y}. \exists \bar{x}. \bigwedge_i \sum_j z_{i,j} x_i^j \doteq y_i).$$

Suppose $\mathbb{C} \not\models \sigma_{n,d}$. Then by completeness of ACF_0 , $\text{ACF}_0 \models \neg \sigma_{n,d}$. Then by compactness, for some $m \in \omega$,

$$\text{ACF} \models \bigwedge_{0 < i < m} \bar{i} \neq 0 \rightarrow \neg \sigma_{n,d}.$$

So if $p > m$ is prime, $\text{ACF}_p \models \neg \sigma_{n,d}$. But this contradicts the Claim. $\square$

5.4.5 Presburger Arithmetik

Beispiel 5.24. (ohne Beweis) In $(\mathbb{Z}; +, <)$ sind $n\mathbb{Z} \subseteq \mathbb{Z}$ ($n \geq 2$) existentiell definierbar aber nicht qf definierbar. Jedoch hat $(\mathbb{Z}; 0, 1, +, -, <, 2\mathbb{Z}, 3\mathbb{Z}, 4\mathbb{Z}, \dots)$ QE.

6 Elementary extensions

Theorem 6.1 (Tarski-Test). *Suppose $\mathcal{M}$ is an $\mathcal{L}$ -structure and A is a subset. TFAE:*

- (i) A is the domain of an elementary substructure;
- (ii) for every $L(A)$ -formula in one free variable $\phi(x)$ gilt: if $\mathcal{M} \models \exists x. \phi(x)$, then $\mathcal{M} \models \phi(a)$ for some $a \in A$.

Proof.

- (i) $\Rightarrow$ (ii): Let $\mathcal{A}$ be the elementary substructure with domain A . Then if $\mathcal{M} \models \exists x. \phi(x)$, then $\mathcal{A} \models \exists x. \phi(x)$ by elementarity; so say $a \in A$ and $\mathcal{A} \models \phi(a)$; but then $\mathcal{M} \models \phi(a)$ by elementarity.
- (ii) $\Rightarrow$ (i): By (ii) with $\phi(x) := x \doteq f(\bar{a})$ we have that A is closed under (≥ 0 -ary) functions. So A is the domain of a substructure $\mathcal{A}$.

We show by induction on complexity that for any $L(A)$ -sentence σ ,

$$\mathcal{A} \models \sigma \Leftrightarrow \mathcal{M} \models \sigma. \quad (1)$$

(1) holds for σ atomic since $\mathcal{A}$ is a substructure, and if it holds for σ and σ' then clearly it holds for $\neg\sigma$ and $(\sigma \wedge \sigma')$.

So suppose $\sigma = \exists x. \phi(x)$, and (1) holds for $\phi(a)$ for any $a \in A$.

If $\mathcal{A} \models \sigma$, then $\mathcal{A} \models \phi(a)$ for some $a \in A$, so $\mathcal{M} \models \phi(a)$ by the induction hypothesis, so $\mathcal{M} \models \sigma$. Conversely, if $\mathcal{M} \models \sigma$, then $\mathcal{M} \models \phi(b)$ for some $b \in \mathcal{M}$, so, by (i), $\mathcal{M} \models \phi(a)$ for some $a \in A$, so $\mathcal{A} \models \phi(a)$ by the induction hypothesis, so $\mathcal{A} \models \sigma$.

□

End of lecture 05

Definition 6.2. An $\mathcal{L}$ -theory T has **built-in Skolem functions**, if for every $\mathcal{L}$ -formula $\phi(x, \bar{y})$ there is $f_{\phi(x, \bar{y})} \in \mathcal{L}$ such that

$$T \models \forall \bar{y}. (\exists x. \phi(x, \bar{y}) \rightarrow \phi(f_{\phi(x, \bar{y})}(\bar{y}), \bar{y})).$$

Lemma 6.3. *If $\mathcal{L}$ has built-in Skolem functions and $A \subseteq \mathcal{M}$ is a subset of an $\mathcal{L}$ -structure, then $\langle A \rangle_{\mathcal{L}}^{\mathcal{M}} \preceq \mathcal{M}$.*

Proof. Let $\mathcal{N} := \langle A \rangle_{\mathcal{L}}^{\mathcal{M}} \leq \mathcal{M}$. Let $\phi(x, \bar{a})$ be an $\mathcal{L}(\mathcal{N})$ -formula, and suppose $\mathcal{M} \models \exists x. \phi(x, \bar{a})$. Then $f_{\phi(x, \bar{y})}(\bar{a}) \in \mathcal{N}$, and $\mathcal{M} \models \phi(f_{\phi(x, \bar{y})}(\bar{a}), \bar{a})$. So by Theorem 6.1 $\mathcal{N} \preceq \mathcal{M}$. □

Lemma 6.4. *Let T be an $\mathcal{L}$ -theory. Then T has a **skolemisation** $T^* \supseteq T$, a theory in a language $\mathcal{L}^* \supseteq \mathcal{L}$ with $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$, such that T^* has built-in Skolem functions, and any model of T expands to a model of T^* .*

Proof. Let $\mathcal{L}_0 := \mathcal{L}$ and $\mathcal{L}_{i+1} := \mathcal{L}_i \cup \{f_{\phi(x, \bar{y})} : \phi(x, \bar{y}) \text{ an } \mathcal{L}_i\text{-formula}\}$ and $\mathcal{L}^* := \bigcup_i \mathcal{L}_i$.

Let

$$T^* := T \cup \{\forall \bar{y}. (\exists x. \phi(x, \bar{y}) \rightarrow \phi(f_{\phi(x, \bar{y})}(\bar{y}), \bar{y})) \mid \phi(x, \bar{y}) \text{ an } \mathcal{L}^*\text{-formula}\}.$$

If $\mathcal{M} \models T$, recursively define the $f_{\phi(x, \bar{y})}$ to witness the existentials (using the axiom of choice) to obtain an expansion to a model of T^* . $\square$

Theorem 6.5 (Löwenheim-Skolem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$ -structure.*

- (i) “Downwards”: *If $A \subseteq \mathcal{M}$ is a subset with $|A| \geq |\mathcal{L}| + \aleph_0$, then there exists an elementary substructure $\mathcal{N} \preceq \mathcal{M}$ containing A , with $|\mathcal{N}| = |A|$.*
- (ii) “Upwards”: *For any cardinal $\kappa \geq |\mathcal{L}| + |\mathcal{M}|$ there exists an elementary extension $\mathcal{N} \succeq \mathcal{M}$ with $|\mathcal{N}| = \kappa$.*

In particular, for any $\kappa \geq |\mathcal{L}| + \aleph_0$, there is $\mathcal{N} \equiv \mathcal{M}$ with $|\mathcal{N}| = \kappa$.

Proof. (i) First, assume $T := \text{Th}(\mathcal{M})$ has built-in skolem functions.

Let $\mathcal{N} := \langle A \rangle_{\mathcal{L}^*}^{\mathcal{M}}$. By Lemma 2.1, $|\mathcal{N}| = |A|$. By Lemma 6.3, $\mathcal{N} \preceq \mathcal{M}$.

Now for the general case, let $\mathcal{L}^*$ and T^* be as in Lemma 6.4, and let $\mathcal{M}^* \models T^*$ be an expansion of $\mathcal{M}$ to $\mathcal{L}^*$. Since $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$, we have $|A| \geq |\mathcal{L}^*| + \aleph_0$, so we obtain $A \subseteq \mathcal{N}^* \preceq \mathcal{M}^*$ with $|\mathcal{N}^*| = |A|$. Then $\mathcal{N} := \mathcal{N}^* \upharpoonright_{\mathcal{L}}$ is as required.

- (ii) Let $\mathcal{L}' := \mathcal{L}(\mathcal{M}) \cup \{c_i : i \in \kappa\}$, where c_i are new constants, and $T' := \text{Th}(\mathcal{M}_{\mathcal{M}}) \cup \{c_i \neq c_j : i \neq j \in \kappa\}$. Then T' is consistent since $\mathcal{M}$ is infinite. Let $\mathcal{M}' \models T'$ with $\mathcal{M} \leq \mathcal{M}'$ and let $A := \{c_i^{\mathcal{M}'} : i \in \kappa\} \subseteq \mathcal{M}'$. Then by (i), there exists $\mathcal{N} \preceq \mathcal{M}'$ with $|\mathcal{N}| = |A| = \kappa$. Now $\mathcal{N} \succeq \mathcal{M}$, since $\mathcal{N} \models \text{Th}(\mathcal{M}_{\mathcal{M}})$. $\square$

Corollary 6.6 (“Skolem’s Paradox”). *If ZFC is consistent, it has a countable model. This is not a paradox!*

Corollary 6.7. *No infinite structure is determined uniquely up to isomorphism by its theory.*

Remark 6.8. In contrast, any *finite* structure is determined uniquely up to isomorphism by its theory.

End of lecture 06

Remark 6.9. $(\mathbb{R}; +, \cdot, <)$ is the unique complete ordered field; but “complete” (every bounded subset has a supremum) is not first-order expressible.

Definition 6.10. Let κ be an infinite cardinal. A theory T is κ -**categorical** if T has a unique model of cardinality κ up to isomorphism.

Theorem 6.11 (Cantor). *DLO is $\aleph_0$ -categorical.*

Proof. (“Back-and-forth argument”)

Let $\mathcal{M}, \mathcal{N} \models \text{DLO}$ with $|\mathcal{M}| = \aleph_0 = |\mathcal{N}|$. Let $\mathcal{M} = (m_i)_{i \in \omega}$ and $\mathcal{N} = (n_i)_{i \in \omega}$. We recursively construct a chain of partial isomorphisms $\theta_i : \mathcal{M} \dashrightarrow \mathcal{N}$ such that

$$|\text{dom } \theta_i| < \aleph_0 \text{ and for all } j < i, \text{ we have } m_j \in \text{dom } \theta_i \text{ and } n_j \in \text{im } \theta_i. \quad (*)$$

Let $\theta_0 := \emptyset$.

Given θ_i satisfying (*),

exactly as in the proof of QE for DLO, θ_i extends to $\theta'_i : \mathcal{M} \dashrightarrow \mathcal{N}$ with $m_i \in \text{dom } \theta'_i$;

similarly, $(\theta'_i)^{-1} : \mathcal{N} \dashrightarrow \mathcal{M}$ extends to $\theta''_i : \mathcal{N} \dashrightarrow \mathcal{M}$ with $n_i \in \text{dom } \theta''_i$;

then $\theta_{i+1} := (\theta''_i)^{-1} : \mathcal{M} \dashrightarrow \mathcal{N}$ satisfies (*).

Then $\theta := \bigcup_i \theta_i : \mathcal{M} \xrightarrow{\cong} \mathcal{N}$ is an isomorphism. $\square$

Theorem 6.12 (Vaughts Criterion). *If an $\mathcal{L}$ -theory T has no finite models and is κ -categorical for some $\kappa \geq |\mathcal{L}| + \aleph_0$, then T is complete.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T$. Both $\mathcal{M}$ and $\mathcal{N}$ are infinite. By Theorem 6.5, there are $\mathcal{M}' \equiv \mathcal{M}$ and $\mathcal{N}' \equiv \mathcal{N}$ with $|\mathcal{M}'| = \kappa = |\mathcal{N}'|$. By κ -categoricity $\mathcal{M}' \cong \mathcal{N}'$. Hence $\mathcal{M} \equiv \mathcal{N}$. $\square$

Notation 6.13. For T a complete $\mathcal{L}$ -theory, we set

$$|T| := |\mathcal{L}| + \aleph_0,$$

being the cardinality of the set of all $\mathcal{L}$ -sentences.

7 Types

Definition 7.1. Let $n \in \omega$, and let $x_1, \dots, x_n$ be a tuple of distinct variables.

- Let $\mathcal{M}$ be an $\mathcal{L}$ -structure. The **type** (in variables $\bar{x}$) of a tuple $\bar{b} \in \mathcal{M}^n$ in $\mathcal{M}$ is

$$\text{tp}^{\mathcal{M}}(\bar{b}) := \{\phi(\bar{x}) : \mathcal{M} \models \phi(\bar{b}); \phi(\bar{x}) \text{ an } \mathcal{L}\text{-formula}\}.$$

- A **type** is the type of some tuple in some structure
- A **partial type** is a subset of type.
- Let T be a consistent $\mathcal{L}$ -theory. The set of **n -types in T** is

$$S_n(T) := \{\text{tp}^{\mathcal{M}}(\bar{b}) : \mathcal{M} \models T; \bar{b} \in \mathcal{M}^n\}.$$

(Technically, this should be written as $S_{\bar{x}}(T)$ as it depends on the choice of variables $\bar{x}$ and not just on n . But “ S_n ” is traditional.)

- If A is a subset of a structure $\mathcal{M}$ and $\bar{b} \in \mathcal{M}^{<\omega}$, the type of $\bar{b}$ **over** A is

$$\text{tp}^{\mathcal{M}}(\bar{b}/A) := \text{tp}^{\mathcal{M}_A}(\bar{b}) \in S_n(\text{Th}(\mathcal{M}_A)).$$

- Let $S_n^{\mathcal{M}}(A) := S_n(\text{Th}(\mathcal{M}_A))$.
- We write $S(T)$ for the set of all types (in arbitrary free variables) in T , and $S^{\mathcal{M}}(A)$ for the set of all types over $A \subseteq \mathcal{M}$.

Lemma 7.2. *A set of $\mathcal{L}$ -formulas $\Phi(\bar{x})$ is a partial type iff it is finitely satisfiable, i.e. for every finite subset $\Phi_0 \subseteq \Phi$ there exist an $\mathcal{L}$ -structure $\mathcal{M}$ such that $\mathcal{M} \models \exists \bar{x}. \bigwedge_{\psi \in \Phi_0} \psi(\bar{x})$.*

A partial type $\Phi(\bar{x})$ is a type iff for every $\mathcal{L}$ -formula $\psi(\bar{x})$, either $\psi(\bar{x}) \in \Phi(\bar{x})$ or $\neg\psi(\bar{x}) \in \Phi(\bar{x})$.

So types in T are precisely maximal consistent (= finitely satisfiable in models of T) sets of formulas in a given tuple of variables.

Proof. Compactness. □

Definition 7.3. $S_n(T)$ is a topological space with basis of open sets $\{[\phi] : \phi(\bar{x}) \text{ an } \mathcal{L}\text{-formula}\}$, where $[\phi] := \{p \in S_n(T) : \phi \in p\} \subseteq S_n(T)$.

Fact 7.4. $S_n(T)$ is a Stone space, i.e. it is compact Hausdorff and totally disconnected.

Proof. ÜA 4. □

Example 7.5 (Types in DLO). DLO has QE, so if $\mathcal{M} \models \text{DLO}$ and $\bar{b} \in \mathcal{M}^{<\omega}$, $\text{tp}^{\mathcal{M}}(\bar{b}/A)$ is determined by the basic $\mathcal{L}_{<}(A)$ -formulas satisfied by $\bar{b}$.

- $S_1(\text{DLO})$: the only consistent basic formula in one variable x is $x = x$, so $|S_1(\text{DLO})| = 1$.
- $S_2(\text{DLO})$ consists of the three types implied (modulo DLO) respectively by $x < y$, $x = y$, and $y < x$.
- $|S_n(\text{DLO})| < \aleph_0$.
- $S_1^{\mathbb{Q}}(\mathbb{Z})$ consists of the types implied by
 - $x = n$ (some $n \in \mathbb{Z}$);
 - $n < x < n + 1$ (some $n \in \mathbb{Z}$);
 - $\{x < n : n \in \mathbb{Z}\}$;
 - $\{x > n : n \in \mathbb{Z}\}$.

End of lecture 07

- More generally, consider $p(x) = S_1(A)$, where $A \subseteq \mathcal{M} \models \text{DLO}$.

If $x = a \in p(x)$ for some $a \in A$, then $x = a$ implies p .

Else, let $L := \{a \in A : a < x \in p\}$ and $R := \{a \in A : a > x \in p\}$. Then (L, R) is a **cut** in A , i.e. $L \dot{\cup} R = A$ and for all $l \in L$ and $r \in R$, we have $l < r$. Then $p(x)$ is implied by $\{l < x : l \in L\} \cup \{x < r : r \in R\}$.

Conversely, if (L, R) is a cut, $\{l < x : l \in L\} \cup \{x < r : r \in R\}$ is finitely satisfiable, so implies a type in $S_1(A)$.

- e.g. $S_1^{\mathbb{Q}}(\mathbb{Q})$ is in bijection with $\mathbb{R} \cup \{-\infty, +\infty\}$ (but the topology is certainly not the Euclidean topology!).

Definition 7.6. For $\Phi(\bar{x})$ a set of $\mathcal{L}$ -formulas and $\bar{c}$ an $|\bar{x}|$ -tuple of constants,

$$\Phi(\bar{c}) := \{\phi(\bar{c}) : \phi(\bar{x}) \in \Phi(\bar{x})\}.$$

If $\bar{y}$ is a $|\bar{x}|$ -tuple of variables, we define $\Phi(\bar{y})$ similarly.

Remark 7.7. Let T be a consistent $\mathcal{L}$ -theory.

- $S_0(T)$ consists of the completions of T .
- The map $S_n(T) \rightarrow S_0(T')$; $p(\bar{x}) \mapsto p(\bar{c})$ is a homeomorphism, where $\bar{c}$ is an n -tuple of new constants and T' is the $\mathcal{L}(\bar{c})$ -theory consisting of the same sentences as T .

Remark 7.8. Let $A \subseteq \mathcal{M}$ be a subset of an $\mathcal{L}$ -structure $\mathcal{M}$.

If $\mathcal{N}$ is another $\mathcal{L}$ -structure containing A and $\mathcal{N}_A \equiv \mathcal{M}_A$ (e.g. if $\mathcal{N} \succeq \mathcal{M}$) then $S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$.

We often just write $S_n(A)$ for $S_n^{\mathcal{M}}(A)$.

Definition 7.9. Let $\Phi(\bar{x})$ a set of formulas.

- If $\bar{a} \in \mathcal{M}$, we write

$$\bar{a} \models \Phi$$

to mean

$$\mathcal{M}_{\bar{a}} \models \Phi(\bar{a}).$$

Then $\bar{a}$ is called a **realisation** of the partial type Φ .

- We write

$$\Phi(\bar{x}) \vdash_T \Phi'(\bar{x})$$

to mean that

$$\Phi(\bar{c}) \vdash_T \Phi'(\bar{c}),$$

where $\bar{c}$ is a $|\bar{x}|$ -tuple of new variables. Equivalently for any $\mathcal{M} \models T$ and $\bar{a} \in \mathcal{M}^{|\bar{x}|}$

$$\bar{a} \models \Phi(\bar{x}) \Rightarrow \bar{a} \models \Phi'(\bar{x}).$$

We write e.g. $\phi \vdash_T \Phi$ to mean $\{\phi\} \vdash_T \Phi$.

Definition 7.10. $\mathcal{M} \models T$ **realises** a set of formulas Φ in T , if some $\bar{b} \in \mathcal{M}^n$ is a realisation of Φ .

If $\mathcal{M}$ does not realise Φ , $\mathcal{M}$ **omits** Φ .

Definition 7.11. A type $p(\bar{x}) \in S_n(T)$ is **isolated** if there is $\phi(\bar{x}) \in p(\bar{x})$ such that $\phi(\bar{x}) \vdash_T p(\bar{x})$. We say then that ϕ **isolates** p .

Lemma 7.12. Let T be a complete theory. Let $p \in S_n(T)$ be isolated and $\mathcal{M} \models T$. Then $\mathcal{M}$ realises p .

Proof. Let $\phi(\bar{x}) \in p(\bar{x})$ with $\phi(\bar{x}) \vdash_T p(\bar{x})$. Since p is finitely satisfiable (by Lemma 7.2) and T is complete, we have $T \models \exists \bar{x}. \phi(\bar{x})$. So say $\bar{b} \in \mathcal{M}^{|\bar{x}|}$ with $\mathcal{M} \models \phi(\bar{b})$. Then $\bar{b} \models p$. $\square$

Example 7.13. Let $K \models \text{ACF}$ and $F \subseteq K$ a subfield. Let $p(x) \in S_1(F)$. Since ACF has QE, p is determined by the polynomial equations over A it implies, i.e. by

$$I_p := \{f(X) \in F[X] : f(x) \doteq 0 \in p\}.$$

This is an ideal: $I_p \trianglelefteq F[X]$. Moreover it is a prime ideal, i.e. $f \cdot g \in I_p \Rightarrow (f \in I_p \text{ or } g \in I_p)$. Indeed, take a realisation $a \in K' \succeq K$; then if $(f \cdot g)(a) = 0$ then $f(a) \cdot g(a) = 0$, so $f(a) = 0$ or $g(a) = 0$ (since K' is an integral domain).

Now $F[X]$ is a principal ideal domain, so $I_p = m_p \cdot F[X]$ for some prime $m_p \in F[X]$. If $m_p = 0$, p is the type of a transcendental element over F , and p might not be realised in K (K could be F^{alg}). Else, p is an *algebraic* type, and it is isolated by $m_p(x) \doteq 0$, and is realised in every algebraically closed field extension of F .

Now consider $p \in S_n(F)$, where $n \geq 1$. As above,

$$I_p := \{f(\bar{X}) \in F[\bar{X}] : f(\bar{x}) \doteq 0 \in p\}$$

is a prime ideal in $F[\bar{X}]$.

Conversely, if $I \trianglelefteq F[\bar{X}]$ is a prime ideal, then $R := F[\bar{X}]/I$ is an integral domain. Let $K' \supseteq F$ be an algebraic closure of the fraction field of R , and let $a_i := X_i/I \in K'$. Let $p := \text{tp}^{K'}(\bar{a}/F)$. Then $I_p = I$.

So $p \mapsto I_p$ is a bijection $S_n(F) \rightarrow \text{Spec}(F[\bar{X}])$, where $\text{Spec}(F[\bar{X}])$ is the set of prime ideals of $F[\bar{X}]$. (This map is continuous if $\text{Spec}(F[\bar{X}])$ is equipped with its usual Zariski topology, but is not a homeomorphism.)

We can also think about this in terms of naive algebraic geometry. If F is a subfield of an algebraically closed field K , a *closed algebraic subset* of K^n over F is the common zero set $V = V(I) \subseteq K^n$ of an ideal $I \trianglelefteq F[\bar{X}]$. V is *irreducible* if it is not the union of two proper closed algebraic subsets over F ; equivalently, if I is a prime ideal. A point $\bar{a} \in V$ is *generic* (over F) if it is contained in no proper closed algebraic subset over F . Then $\text{tp}(\bar{a}/F) = p_{I_p}$, and conversely any $p \in S_n(F)$ is of this form for some $\bar{a} \in K^n$ for some K (e.g. by considering $F[\bar{X}]/I_p$ as above). In other words: the types in ACF are precisely the types of generics of irreducible closed algebraic sets.

Finally, consider an arbitrary subset $A \subseteq K$. Let $F \leq K$ be the subfield generated by A . Then a type over A determines a type over F . In other words, the restriction map $S_n(F) \rightarrow S_n(A)$ is a bijection.

End of lecture 08

7.1 Saturation

Lemma 7.14 (“Joint Consistency for Constants”). *Let T be a complete $\mathcal{L}$ -theory. For $i \in I$ let C_i be a set of constants, and suppose $C_i \cap C_j = \emptyset = C_i \cap \mathcal{L}$ for $i \neq j$. Let $T_i \supseteq T$ be a consistent $\mathcal{L} \cup C_i$ -theory. Then $\bigcup_{i \in I} T_i$ is consistent.*

Remark 7.15. In fact this holds if we add new relations and functions too. This is known as “Robinson’s Joint Consistency Theorem” (a proof can be found in Chang&Keisler).

Proof. If $\bigcup_{i \in I} T_i$ is inconsistent then by compactness $T \cup \{\phi_i(\bar{c}_i) : i \in I_0\}$ is inconsistent where $I_0 \subseteq I$ is finite and $T_i \models \phi_i(\bar{c}_i)$ and $\bar{c}_i \in C_i^{<\omega}$. WLOG $I_0 = \{1, 2, \dots, n\}$.

But then $T \models \forall \bar{x}_1, \dots, \bar{x}_n. \neg \bigwedge_{1 \leq i \leq n} \phi_i(\bar{x}_i)$ with the $\bar{x}_i$ disjoint tuples.

Then $T \models \forall \bar{x}_1, \dots, \bar{x}_n. \bigvee_{1 \leq i \leq n} \neg \phi_i(\bar{x}_i)$.

Then $T \models \bigvee_{1 \leq i \leq n} \forall \bar{x}_i. \neg \phi_i(\bar{x}_i)$. So $T \cup \{\phi_i(\bar{c}_i)\}$ is inconsistent for some i . This contradicts consistency of T_i . $\square$

Lemma 7.16. *Given $\mathcal{M}$ an infinite $\mathcal{L}$ -structure and $A \subseteq \mathcal{M}$, there exists $\mathcal{N} \succeq \mathcal{M}$ which realises every $p \in S(A)$.*

Proof. For each $n \geq 1$ and each n -type $p \in S(A)$, let $\bar{c}_p$ be a new tuple of constants with $|\bar{c}_p| = n$.

We must show $T' := \text{Th}(\mathcal{M}_{\mathcal{M}}) \cup \bigcup_{p \in S(A)} p(\bar{c}_p)$ is consistent.

By definition of $S(A)$, each $\text{Th}(\mathcal{M}_A) \cup p(\bar{c}_p)$ is consistent. Also $\text{Th}(\mathcal{M}_{\mathcal{M}}) \supseteq \text{Th}(\mathcal{M}_A)$ is consistent. So T' is consistent by Lemma 7.14. $\square$

Definition 7.17. Let κ be an infinite cardinal. An $\mathcal{L}$ -structure $\mathcal{M}$ is κ -**saturated** if for any $A \subseteq \mathcal{M}$ with $|A| < \kappa$ every $p \in S_1(A)$ is realised in $\mathcal{M}$.

$\mathcal{M}$ is **saturated**, if it is $|\mathcal{M}|$ -saturated.

Example 7.18. $(\mathbb{Q}; <)$ is $\aleph_0$ -saturated.

$\mathbb{Q}^{\text{alg}} \models \text{ACF}$ is not $\aleph_0$ -saturated (since it omits the transcendental type in $S_1(\emptyset)$).

Lemma 7.19. *If $\mathcal{M}$ is κ -saturated, for any $A \subseteq \mathcal{M}$ with $|A| < \kappa$ and every $n \geq 1$, every $p \in S_n(A)$ is realised in $\mathcal{M}$.*

Proof. By induction on n .

Sei $p(x_1, \dots, x_n, y) \in S_{n+1}(A)$. Setze $q(x_1, \dots, x_n) := \{\phi(x_1, \dots, x_n) : \phi \in p\} \in S_n(A)$. Inductively, q is realised in $\mathcal{M}$. Say $\bar{a} \in \mathcal{M}^n$ with $\bar{a} \models q$.

Now $p(\bar{a}, y) = \{\phi(\bar{a}, y) : \phi \in p\} \in S_1^{\mathcal{M}}(A \cup \{a_1, \dots, a_n\})$; Indeed, If $\Phi_0(\bar{x}, y) \subseteq_{\text{fin}} p(\bar{x}, y)$ is a finite subset, Since $\text{tp}^{\mathcal{M}}(a/A) = q(\bar{x}) \ni \exists y. \bigwedge_{\phi \in \Phi_0} \phi(\bar{x}, y)$, we have $\mathcal{M} \models \exists y. \bigwedge_{\phi \in \Phi_0} \phi(\bar{a}, y)$.

Hence (since $|A| + n < \kappa$) some $b \in \mathcal{M}$ with $b \models p(\bar{a}, y)$ and then $(\bar{a}, b) \models p(\bar{x}, y)$. $\square$

Lemma 7.20. *Suppose $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ is partial elementary and $A \subseteq \text{dom } \theta$ and $p(\bar{x}) \in S^{\mathcal{M}}(A)$.*

(i) *The **conjugate** of p by θ*

$$p^\theta(\bar{x}) := \{\phi(\bar{x}, \theta(\bar{a})) : \phi(\bar{x}, \bar{a}) \in p(\bar{x}); \phi \text{ a } \mathcal{L}\text{-formula}\}$$

is a type $p^\theta \in S^{\mathcal{N}}(\theta(A))$.

(ii) *For $b \in \mathcal{M}$, an extension θ' of θ with $\text{dom } \theta' = \text{dom } \theta \cup \{b\}$ is partial elementary iff $\theta'(b) \models \text{tp}(b/\text{dom } \theta)^\theta$.*

Proof. (i) For a finite subset $\Phi_0(\bar{x}) \subseteq_{\text{fin}} p(\bar{x})$, write $\bigwedge \Phi_0$ as $\psi(\bar{x}, \bar{a})$, where ψ is an $\mathcal{L}$ -formula and $\bar{a} \in A^{<\omega}$. Then $\mathcal{M} \models \exists \bar{x}. \psi(\bar{x}, \bar{a})$. So by elementarity, $\mathcal{N} \models \exists \bar{x}. \psi(\bar{x}, \theta(\bar{a}))$.

(ii) Immediate. $\square$

Lemma 7.21. *If $\mathcal{M} \equiv \mathcal{N}$ and $|\mathcal{M}| = |\mathcal{N}| \geq \aleph_0$ and $\mathcal{M}$ and $\mathcal{N}$ are both saturated, then $\mathcal{M} \cong \mathcal{N}$.*

Proof. Back-and-forth.

Let $\mathcal{M} = (m_\alpha)_{\alpha \in \lambda}$ and $\mathcal{N} = (n_\alpha)_{\alpha \in \lambda}$. We recursively construct a chain of partial isomorphisms $\theta_\alpha : \mathcal{M} \dashrightarrow \mathcal{N}$ for $\alpha \in \lambda$ such that

$$|\text{dom } \theta_\alpha| \leq 2 \cdot |\alpha| \text{ und for all } \beta < \alpha, \text{ we have } m_\beta \in \text{dom } \theta_\alpha \text{ and } n_\beta \in \text{im } \theta_\alpha. \quad (*)$$

Let $\theta_0 := \emptyset$. For η a limit ordinal, let $\theta_\eta := \bigcup_{\alpha < \eta} \theta_\alpha$. We have $|\text{dom } \theta_\eta| \leq |\eta| = 2 \cdot |\eta|$.

Given θ_α satisfying $(*)$,

By saturation, $\text{tp}(m_\alpha / \text{dom } \theta_\alpha)^{\theta_\alpha} \in S_1(\text{im } \theta_\alpha)$ is realised in $\mathcal{N}$.

Hence θ_α extends to $\theta'_\alpha : \mathcal{M} \dashrightarrow \mathcal{N}$ with $m_\alpha \in \text{dom } \theta'_\alpha$;

symmetrically, $(\theta'_\alpha)^{-1} : \mathcal{N} \dashrightarrow \mathcal{M}$ extends to $\theta''_\alpha : \mathcal{N} \dashrightarrow \mathcal{M}$ with $n_\alpha \in \text{dom } \theta''_\alpha$; then $\theta_{\alpha+1} := (\theta''_\alpha)^{-1} : \mathcal{M} \dashrightarrow \mathcal{N}$ satisfies $(*)$.

Then $\theta := \bigcup_\alpha \theta_\alpha : \mathcal{M} \xrightarrow{\cong} \mathcal{N}$ is an isomorphism. $\square$

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Definition 7.22. An $\mathcal{L}$ -structure $\mathcal{M}$ is κ -universal if any $\mathcal{N} \equiv \mathcal{M}$ with $|\mathcal{N}| < \kappa$ elementarily embeds in $\mathcal{M}$.

Lemma 7.23. *Let $\mathcal{M}$ be an κ -saturated $\mathcal{L}$ -structure. Then $\mathcal{M}$ is κ^+ -universal.*

Proof. Suppose $\mathcal{N} \equiv \mathcal{M}$ with $\lambda := |\mathcal{N}| < \kappa^+$. Say $\mathcal{N} = \{a_\alpha : \alpha \in \lambda\}$. We build a chain of partial elementary maps $\theta_\alpha : \mathcal{N} \dashrightarrow \mathcal{M}$ with $\text{dom } \theta_\alpha = A_\alpha := \{a_\beta : \beta < \alpha\}$.

Set $\theta_0 := \emptyset$. For η a limit ordinal, set $\theta_\eta := \bigcup_{\alpha < \eta} \theta_\alpha$.

Since $|A_\alpha| = |\alpha| < \lambda \leq \kappa$, by κ -saturation, $\text{tp}(a_\alpha / A_\alpha)^{\theta_\alpha}$ is realised in $\mathcal{M}$. Set $\theta_{\alpha+1}(a_\alpha)$ to be a realisation. $\square$

8 Countable models of countable theories

8.1 Countable saturated models

Lemma 8.1 (Tarski's Chain Lemma). *If $(I; <)$ is a linear order and $(\mathcal{M}_i)_{i \in I}$ is an elementary chain, meaning $\mathcal{M}_i \preceq \mathcal{M}_j$ for $i < j$, then $\mathcal{M}_i \preceq \bigcup_{i \in I} \mathcal{M}_i$ for all i .*

Proof. ÜA 1.1(b). $\square$

Definition 8.2. A theory T is **small** if $|S_n(T)| \leq \aleph_0$ for all $n \in \omega$.

Example 8.3. $(\mathbb{Q}; <)$ ist schmal. $(\mathbb{Q}; <)_{\mathbb{Q}}$ ist nicht schmal.

Theorem 8.4. *Let T be a countable (i.e. $|\mathcal{L}| \leq \aleph_0$) complete $\mathcal{L}$ -theory with infinite models.*

Then T has a countable saturated model iff T is small.

Proof. $\Rightarrow$: Let $n \in \omega$. Every type in $S_n(T)$ is realised in the countable saturated model, so $|S_n(T)| \leq \aleph_0$.

$\Leftarrow$: If $A \subseteq_{\text{fin}} \mathcal{M} \models T$, then $|S_1(A)| \leq |S_{|A|+1}(T)| \leq \aleph_0$; indeed, if $A = \{a_1, \dots, a_n\}$ then $p(x, \bar{a}) \mapsto p(x, \bar{y})$ is an injection $S_1(A) \hookrightarrow S_{n+1}(T)$.

We build an elementary chain $(\mathcal{M}_i)_{i \in \omega}$ of countable models. Let $\mathcal{M}_0 \models T$ with $|\mathcal{M}_0| = \aleph_0$, which exists by Löwenheim-Skolem. Given, let $X := \bigcup_{A \subseteq_{\text{fin}} \mathcal{M}_i} S_1(A)$. Then $|X| \leq \aleph_0$. So by Lemma 7.16 and Löwenheim-Skolem, there is a countable model $\mathcal{M}_{i+1} \succeq \mathcal{M}_i$ which realises all $p \in X$.

Now let $\mathcal{M} := \bigcup_{i \in \omega} \mathcal{M}_i \models T$. Then $|\mathcal{M}| \leq \aleph_0$ and if $A \subseteq_{\text{fin}} \mathcal{M}$, then $A \subseteq_{\text{fin}} \mathcal{M}_i$ for some $i \in \omega$. Hence every $p \in S_1(A)$ is realised in $\mathcal{M}_{i+1}$ and hence in $\mathcal{M} \succeq \mathcal{M}_{i+1}$. $\square$

8.2 Omitting types

Definition 8.5. Let T be a consistent $\mathcal{L}$ -theory.

- A $\mathcal{L}$ -formula $\phi(\bar{x})$ is **consistent** (with T), if $T \not\models \neg \exists x. \psi(\bar{x})$.
- A set of formulas $\Phi(\bar{x})$ in T is **isolated** if there exists a consistent $\phi(\bar{x})$ s.t. $\phi(\bar{x}) \vdash_T \Phi(\bar{x})$.

Theorem 8.6 (Omitting Types Theorem). *Let T be a countable consistent theory.*

Let $(\Phi_k(\bar{x}_k))_{k \in \omega}$ be non-isolated sets of formulas. Then there exists a countable model $\mathcal{M} \models T$ which omits every Φ_k .

Proof. Let $C = \{c_i : i < \omega\}$ be a set of new constants.

Enumerate the $\mathcal{L}(C)$ -formulas in x as $(\psi_i(x) : i < \omega)$.

Let $\xi : \{(k, \bar{c}) : k \in \omega; \bar{c} \in C^{|\bar{x}_k|}\} \rightarrow \omega$ be a bijection.

We construct an increasing chain $(\Sigma_i)_{i \in \omega}$ of sets of $\mathcal{L}(C)$ -sentences such that

- (i) $|\Sigma_i| < \aleph_0$;
- (ii) $T_i := T \cup \Sigma_i$ is consistent;
- (iii) if $j < i$, there is $c \in C$ such that $T_i \models \exists x. \psi_j(x) \rightarrow \psi_j(c)$;
- (iv) if $\xi(k, \bar{c}) < i$, there is $\phi(\bar{x}_k) \in \Phi_k(\bar{x}_k)$ such that $T_i \models \neg \phi(\bar{c})$.

Let $\Sigma_0 := \emptyset$. Suppose Σ_i satisfies (i)-(iv).

Say $c \in C$ does not appear in Σ_i nor in ψ_i , and let $\Sigma'_{i+1} := \Sigma_i \cup \{\exists x. \psi_i(x) \rightarrow \psi_i(c)\}$. Note that $T \cup \Sigma'_{i+1}$ is consistent.

Say $\xi(k, \bar{c}) = i$, and let $\bar{x} := \bar{x}_k$. Let $\delta(\bar{x}, \bar{y})$ be such that $\bigwedge \Sigma'_{i+1} = \delta(\bar{c}, \bar{c}')$, where $\bar{c}' \in (C \setminus \{c_1, \dots, c_{|\bar{x}|}\})^{<\omega}$. Now $T \not\models \neg \exists \bar{x}. \exists \bar{y}. \delta(\bar{x}, \bar{y})$, so by non-isolation of Φ_k there is $\phi_i(\bar{x}) \in \Phi_k(\bar{x}_k)$ such that

$$\exists \bar{y}. \delta(\bar{x}, \bar{y}) \not\models_T \phi_i(\bar{x}). \quad (*)$$

Let $\Sigma_{i+1} := \Sigma'_{i+1} \cup \{\neg \phi_i(\bar{c})\}$. Then $T \cup \Sigma_{i+1}$ is consistent by (*).

Now let $T_\omega := T \cup \bigcup_{i \in \omega} \Sigma_i$. This is consistent by (ii) (since $(\Sigma_i)_i$ is an increasing chain), so let $\mathcal{M} \models T_\omega$. By (iii) and Tarski's test, $\{c^{\mathcal{M}} : c \in C\}$ is the domain of a countable elementary substructure $\mathcal{N} \preceq \mathcal{M}$. By (iv), $\mathcal{N}$ omits each Φ_k . $\square$

8.3 Prime models

Definition 8.7. $\mathcal{M} \models T$ is a **prime** model of a theory T if it elementarily embeds in any model of T .

Example 8.8. $(\mathbb{Z}; S)$ is a prime model of its theory.

$(\mathbb{Q}; <)$ is a prime model of DLO.

Definition 8.9. An $\mathcal{L}$ -structure $\mathcal{M}$ is **atomic** if $\text{tp}(\bar{a})$ is isolated for every $\bar{a} \in \mathcal{M}^{<\omega}$.

Definition 8.10. A formula $\phi(\bar{x})$ is an **atom** modulo T if it is consistent with T and isolates a type in T ; equivalently, for no $\psi(\bar{x})$ are both $\phi \wedge \psi$ and $\phi \wedge \neg\psi$ consistent with T .

Notation 8.11. $\bar{a}\bar{b} := (a_1, \dots, a_{|\bar{a}|}, b_1, \dots, b_{|\bar{b}|})$.

Lemma 8.12 (“Monotonicity and transitivity of isolation”). *Let $\bar{a}, \bar{b} \in \mathcal{M}^{<\omega}$. Then $\text{tp}(\bar{a}\bar{b})$ is isolated iff $\text{tp}(\bar{a}/\bar{b})$ and $\text{tp}(\bar{b})$ are isolated.*

Proof. $\Rightarrow$: Say $\phi(\bar{x}, \bar{y})$ isolates $\text{tp}(\bar{a}\bar{b})$. Then

- $\phi(\bar{x}, \bar{b})$ isolates $\text{tp}(\bar{a}/\bar{b})$; indeed, if $\bar{a} \models \psi(\bar{x}, \bar{b})$ then $\bar{a}\bar{b} \models \psi(\bar{x}, \bar{y})$, and so $\phi(\bar{x}, \bar{y}) \vdash_T \psi(\bar{x}, \bar{y})$, and so $\phi(\bar{x}, \bar{b}) \vdash_{T_{\bar{b}}} \psi(\bar{x}, \bar{b})$.
- $\exists \bar{x}. \phi(\bar{x}, \bar{y})$ isolates $\text{tp}(\bar{b})$; indeed, if $\bar{b} \models \psi(\bar{y})$ then $\phi(\bar{x}, \bar{y}) \vdash_T \psi(\bar{y})$, and so $\exists \bar{x}. \phi(\bar{x}, \bar{y}) \vdash_T \psi(\bar{y})$.

$\Leftarrow$: Say $\phi(\bar{y})$ isolates $\text{tp}(\bar{b})$ and $\xi(\bar{x}, \bar{b})$ isolates $\text{tp}(\bar{a}/\bar{b})$ (where $\xi(\bar{x}, \bar{y})$ is an $\mathcal{L}$ -formula).

Then $\xi(\bar{x}, \bar{y}) \wedge \phi(\bar{y})$ isolates $\text{tp}(\bar{a}\bar{b})$. Indeed, if $\bar{a}\bar{b} \models \psi(\bar{x}, \bar{y})$ then $\bar{a} \models \psi(\bar{x}, \bar{b})$, hence $\xi(\bar{x}, \bar{b}) \vdash_{T_{\bar{b}}} \psi(\bar{x}, \bar{b})$, hence $\bar{b} \models \forall \bar{x}. (\xi(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y}))$, hence $T \models \forall \bar{y}. (\phi(\bar{y}) \rightarrow \forall \bar{x}. (\xi(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y})))$, hence $T \models \forall \bar{x}, \bar{y}. ((\xi(\bar{x}, \bar{y}) \wedge \phi(\bar{y})) \rightarrow \psi(\bar{x}, \bar{y}))$. $\square$

Lemma 8.13. *Let $\mathcal{M}$ be an infinite $\mathcal{L}$ -structure, where $|\mathcal{L}| \leq \aleph_0$.*

Then $\mathcal{M}$ is a prime model of $\text{Th}(\mathcal{M})$ iff $\mathcal{M}$ is countable and atomic.

Proof. Suppose $\mathcal{M} \models T$ is prime. Then $\mathcal{M}$ is countable since it embeds in a countable model, by Löwenheim-Skolem. Let $\bar{a} \in \mathcal{M}^{<\omega}$. Then $\text{tp}(\bar{a})$ is realised in any $\mathcal{M}' \models T$ (namely by $\theta(\bar{a})$, where $\theta : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$). So by the omitting types theorem, $\text{tp}(\bar{a})$ is isolated. Hence $\mathcal{M}$ is atomic.

Conversely, suppose $\mathcal{M} = (a_i)_{i \in \omega}$ is countable atomic, and let $\mathcal{M}' \models T$. We build a chain of partial elementary maps $\theta_i : \mathcal{M} \rightarrow \mathcal{M}'$ with $\text{dom } \theta_i = \{a_j : j < i\}$.

Set $\theta_0 := \emptyset$.

Given θ_i , $p_i := \text{tp}(a_i/a_0, \dots, a_{i-1})$ is isolated by atomicity (and Lemma 8.12), and hence $p_i^{\theta_i}$ is isolated and hence realised by some $b_i \in \mathcal{M}'$. Let $\theta_{i+1}(a_i) := b_i$. Then θ_{i+1} is partial elementary (by Lemma 7.20(ii)).

Then $\bigcup_{i < \omega} \theta_i : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ is an elementary embedding. So $\mathcal{M}$ is prime. $\square$

Lemma 8.14. *Let $\mathcal{M} \equiv \mathcal{N}$ be countable atomic elementarily equivalent structures. Then $\mathcal{M}$ is isomorphic to $\mathcal{N}$.*

Proof. $\ddot{U}A$. $\square$

Proposition 8.15. *Let T be a countable theory. Then T has at most one prime model up to isomorphism.*

Proof. $\ddot{U}A$ (it follows from Lemma 8.14 and Lemma 8.13). $\square$

Definition 8.16. We say the **isolated types are dense in $S(T)$** if for each formula $\phi(\bar{x})$ consistent with T , there is some isolated $p(\bar{x}) \in S(T)$ with $\phi \in p$.

Theorem 8.17. *A countable complete theory T has a prime model iff isolated types are dense in $S(T)$.*

Proof. $\Rightarrow$: Say (by Lemma 8.13) $M \models T$ is countable atomic. Then any consistent $\phi(\bar{x})$ has a realisation in $\mathcal{M}$ (since T is complete), which by atomicity has isolated type.

$\Leftarrow$: Let $n \in \omega$ and $\Psi_n(x_1, \dots, x_n) := \{\neg\psi(\bar{x}) : \psi(\bar{x}) \text{ an atom}\}$. Suppose a formula $\phi(\bar{x})$ isolates $\Psi_n(\bar{x})$. By density of isolated types, $\psi(\bar{x}) \vdash_T \phi(\bar{x})$ for some atom $\psi(\bar{x})$, but then $\psi(\bar{x}) \vdash_T \neg\psi(\bar{x})$, contradicting consistency of $\psi(\bar{x})$. So each $\Psi_n(\bar{x})$ is not isolated, so by the omitting types theorem, there is a countable $\mathcal{M} \models T$ which omits each Ψ_n . Then $\mathcal{M}$ is atomic: if $\bar{a} \in \mathcal{M}^{<\omega}$, there is an atom $\psi(\bar{x})$, such that $\bar{a} \not\models \neg\psi(\bar{x})$, hence $\bar{a} \models \psi(\bar{x})$, hence $\psi(\bar{x})$ isolates $\text{tp}(\bar{a})$. $\square$

End of lecture 11

Notation 8.18. We write elements of $2^{<\omega}$ or 2^ω as binary strings, with $\emptyset$ for the empty string.

$s < t$ means that s is a prefix of t , i.e. $t = st'$ for some t' .

Definition 8.19. A **binary tree of formulas** for a theory T is a family of formulas $(\phi_s(\bar{x}))_{s \in 2^{<\omega}}$ such that for each $s \in 2^{<\omega}$:

- $\phi_s(\bar{x})$ is consistent with T ;
- $\phi_{s0}(\bar{x}) \vdash_T \phi_s(\bar{x})$ and $\phi_{s1}(\bar{x}) \vdash_T \phi_s(\bar{x})$;
- $\phi_{s0}(\bar{x}) \vdash_T \neg\phi_{s1}(\bar{x})$.

Lemma 8.20. *If a countable theory T has a binary tree of formulas, then $S(T) = 2^{\aleph_0}$.*

Proof. Say $(\phi_s(\bar{x}))_{s \in 2^{<\omega}}$ is a binary tree and $|\bar{x}| = n$. For $t \in 2^\omega$, $\{P_s(\bar{x}) : s < t\}$ is consistent, so extends to a type $p_t(\bar{x}) \in S_n(T)$. Then if $t \neq t'$, there is $s \in 2^{<\omega}$ such that $s0 < t$ and $s1 < t'$ or the other way round, so $p_t(\bar{x}) \neq p_{t'}(\bar{x})$.

So $|S(T)| \geq |S_n(T)| \geq |2^\omega| = 2^{\aleph_0}$.

Since the language $\mathcal{L}$ of T is countable, $|S(T)| \leq |\mathcal{P}(\{\mathcal{L}\text{-formulas}\})| = 2^{\aleph_0}$. $\square$

Example 8.21. Consider 2^ω as a structure in the language $\{P_s : s \in 2^{<\omega}\}$, where $P_s(2^\omega) := \{t : s < t\}$, and let T_{BT} be its theory.

Then $\{P_s(x) : s \in 2^{<\omega}\}$ is a binary tree of formulas in T_{BT} .

Exercise: T_{BT} has QE. It follows that $2^\omega \ni t \mapsto \text{tp}(t) \in S_1(T_{BT})$ is a bijection, and none of these types are isolated.

So T_{BT} is not small, and isolated types are not dense, and there is no prime model.

Bonus exercise: explicitly describe a countable model.

Lemma 8.22. *Let T be a consistent theory.*

- (i) *If the isolated types are not dense in $S(T)$, then T has a binary tree of formulas.*
- (ii) *If T is small, the isolated types are dense in $S(T)$.*

Proof. (i) Say $\phi_\emptyset(\bar{x})$ is consistent and in no isolated type.

Given $\phi_s(\bar{x})$ which is consistent and in no isolated type ($s \in 2^{<\omega}$), $\phi_s(\bar{x})$ is not an atom, so there is $\psi(\bar{x})$ such that $\phi_{s0} := \phi_s(\bar{x}) \wedge \neg\psi(\bar{x})$ and $\phi_{s1} := \phi_s(\bar{x}) \wedge \psi(\bar{x})$ are consistent, and each is in no isolated type since ϕ has this property.

So we may recursively construct a binary tree of formulas $(\phi_s(\bar{x}))_{s \in 2^{<\omega}}$.

- (ii) Follows from (i) and Lemma 8.20. □

Proposition 8.23. *Any countable complete small theory has a prime model.*

Proof. Immediate from Lemma 8.22(ii) and Theorem 8.17 □

Remark 8.24. The converse fails; consider $(\mathbb{Q}; <)_{\mathbb{Q}}$ ($\ddot{U}A$).

Example 8.25. Let k be a countable field. The theory of infinite k -vector spaces is complete, countable, and small. The countable models are the vector spaces V_d of dimensions $d \in (\omega \setminus 0) \cup \{\aleph_0\}$. V_1 is the prime model, and $V_{\aleph_0}$ is the countable saturated model.

Proposition 8.26. *For a countable theory T , TFAE:*

- (i) *T is not small, i.e. $|S(T)| > \aleph_0$;*
- (ii) *T has a binary tree of formulas;*
- (iii) *$|S(T)| = 2^{\aleph_0}$.*

Furthermore, if T is not small, then

- (iv) *T has $2^{\aleph_0}$ countable models up to isomorphism.*

Proof. $\ddot{U}A$. □

Remark 8.27. There do exist small theories with $2^{\aleph_0}$ countable models, e.g. $\text{Th}((\omega \times \omega; (P_i)_i)_{\omega \times \omega})$, where $P_i(\omega \times \omega) = \{i\} \times \omega$.

Corollary 8.28. *If a countable theory T has countably many countable models, then T is small, and hence T has a prime countable model and a saturated countable model.*

Conjecture 8.29 (Vaught). *If T is a countable theory with uncountably many countable models, it has $2^{\aleph_0}$ countable models.*

(Note: it's easy to see that a countable theory has at most $2^{\aleph_0}$ countable models, so this is immediate if we assume CH.)

8.4 Ryll-Nardzewski

Theorem 8.30 (Ryll-Nardzewski). *Let T be a complete countable $\mathcal{L}$ -theory with infinite models. TFAE:*

- (A) T is $\aleph_0$ -categorical.
- (B1) For all $n \in \omega$ and $\mathcal{M} \models T$, there are only finitely many definable subsets of $\mathcal{M}^n$.
- (B1') For all $n \in \omega$, we have $|\Phi_{n,T}| < \aleph_0$, where $\Phi_{n,T} := \{\phi(x_1, \dots, x_n)\} / \leftrightarrow_T$ is the set of $(\leftrightarrow_T)$ -equivalence classes of $\mathcal{L}$ -formulas $\phi(x_1, \dots, x_n)$.
- (B2) Every type in T is isolated.
- (B3) For all $n \in \omega$, we have $|S_n(T)| < \aleph_0$.
- (C) Every countable model of T is saturated.
- (D) Every countable model of T is prime.
- (E) T has a countable model which is saturated and prime.

(The equivalence (A) $\Leftrightarrow$ (B1) is the key result here, and is what is most often referred to as the Ryll-Nardzewski Theorem.)

Proof.

- (B1) $\Leftrightarrow$ (B1') If $\mathcal{M} \models T$, then $\phi(\bar{x}) \leftrightarrow_T \psi(\bar{x})$ iff $\phi(\mathcal{M}) = \psi(\mathcal{M})$.
- (B1') $\Rightarrow$ (B2) Given $p(\bar{x}) \in S(T)$, $p(\bar{x})$ contains by (B1') only finitely many formulas $\phi_1(\bar{x}), \dots, \phi_k(\bar{x})$ up to equivalence. Then $\bigwedge_i \phi_i \in p$ isolates p .
- (B2) $\Rightarrow$ (B3) Suppose each $p(\bar{x}) \in S(T)$ is isolated, say by $\phi_p(\bar{x})$. Let $n \in \omega$.
Suppose $|S_n(T)|$ is infinite. Then $\{\neg\phi_p(\bar{x}) : p \in S_n(T)\}$ is finitely satisfiable and so can be completed to some $p \in S_n(T)$, but then $\phi_p \vdash_T p \supset \neg\phi_p$, contradicting consistency of ϕ_p .
- (B3) $\Rightarrow$ (B1') The map $\phi(x_1, \dots, x_n) \mapsto \{p \in S_n(T) : \phi(\bar{x}) \in p(\bar{x})\}$ induces an injection $\Phi_{n,T} \hookrightarrow \mathcal{P}(S_n(T))$; indeed, if $\phi(\bar{x}) \not\leftrightarrow_T \psi(\bar{x})$, then $T \models \exists \bar{x}. \neg(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}))$, so for some $p \in S_n(T)$, we have $\phi \in p \not\leftrightarrow \psi \in p$. So $\Phi_{n,T}$ is finite if $S_n(T)$ is.

For the remaining equivalences, note first that since T is complete and has infinite models, it has no finite models, so countable models have cardinality $\aleph_0$.

- (A) $\Rightarrow$ (D) By Löwenheim-Skolem, every model of T has a countable elementary submodel. So if $\mathcal{M}$ is the unique countable model, it elementarily embeds in every model, so is prime.
- (D) $\Rightarrow$ (B2) Every type is (by Löwenheim-Skolem) realised in a countable model, which by (D) and Lemma 8.13 is atomic.
- (B2) $\Rightarrow$ (C) Let $\mathcal{M} \models T$ be countable, and let $A \subseteq_{\text{fin}} \mathcal{M}$. Then every $p \in S(A)$ is isolated by (B2) (and Lemma 8.12), and hence realised in $\mathcal{M}$. So $\mathcal{M}$ is saturated.
- (C) $\Rightarrow$ (A) Lemma 7.21.

((C) $\wedge$ (D)) $\Rightarrow$ (E) A countable model exists by Löwenheim-Skolem, so this is immediate.

(E) $\Rightarrow$ (D) If $\mathcal{M} \models T$ is a countable saturated prime model and $\mathcal{N} \models T$ is countable, then $\mathcal{N} \xrightarrow{\prec} \mathcal{M}$ by Lemma 7.23. Then $\mathcal{N}$ is prime since $\mathcal{M}$ is. .

□

Remark 8.31. We can also give direct proofs of some of the other implications:

(A) $\Rightarrow$ (B) If some type is not isolated, by the omitting types theorem we have a countable model which omits it, but we also have a countable model which realises it.

(B2) $\Rightarrow$ (D) Lemma 8.13.

(D) $\Rightarrow$ (A) Proposition 8.15.

((A) $\wedge$ (B)) $\Rightarrow$ (E) By (B), T is small, so saturated and prime countable models exist; by (A) they are isomorphic.

(E) $\Rightarrow$ (B2) By (E), an atomic model realises every type.

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8.5 Fraïssé constructions

Let $\mathcal{L}$ be finite and *relational*, i.e. containing only relation symbols. In this section, we consider the empty $\mathcal{L}$ -structure to be an $\mathcal{L}$ -structure.

Our aim is to find $\aleph_0$ -categorical theories.

Remark 8.32. Let $n \in \omega$. There are up to isomorphism only finitely many $\mathcal{L}$ -structures of cardinality n .

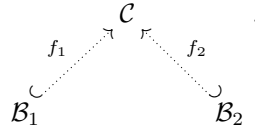
Definition 8.33. The **age** of an $\mathcal{L}$ -structure $\mathcal{M}$ is the class of finite $\mathcal{L}$ -structures which embed in $\mathcal{M}$,

$$\text{age}(\mathcal{M}) := \{\mathcal{A} : |\mathcal{A}| < \aleph_0; \exists f : \mathcal{A} \hookrightarrow \mathcal{M}\} = \{\mathcal{A} : \mathcal{A} \cong \mathcal{A}' \leq_{\text{fin}} \mathcal{M}\}.$$

Lemma 8.34. Any age $\mathcal{K}$ satisfies:

(HP) If $\mathcal{A} \in \mathcal{K}$, then $\text{age}(\mathcal{A}) \subseteq \mathcal{K}$;

(JEP) If $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{K}$, then there exist $\mathcal{C} \in \mathcal{K}$ and embeddings $f_i : \mathcal{B}_i \hookrightarrow \mathcal{C}$.



Proof. (HP) Clear.

(JEP) Say $f_i : \mathcal{B}_i \hookrightarrow \mathcal{M}$. Let $\mathcal{C} := \langle f_1(\mathcal{B}_1) \cup f_2(\mathcal{B}_2) \rangle^{\mathcal{M}} \leq \mathcal{M}$. Then $f_i : \mathcal{B}_i \hookrightarrow \mathcal{C} \in \mathcal{K}$.

□

Conversely:

Lemma 8.35. *Any non-empty class $\mathcal{K}$ of finite $\mathcal{L}$ -structures satisfying (HP) and (JEP) is the age of a countable $\mathcal{L}$ -structure.*

Proof. By Remark 8.32 we can find $\mathcal{A}_i \in \mathcal{K}$ for $i \in \omega$ such that any $\mathcal{A} \in \mathcal{K}$ is isomorphic to some $\mathcal{A}_i$.

We construct a countable chain $\mathcal{D}_0 \leq \mathcal{D}_1 \leq \dots$ with $\mathcal{D}_i \in \mathcal{K}$, such that each $\mathcal{A}_j$ for $j < i$ embeds in $\mathcal{D}_i$.

Let $\mathcal{D}_0 := \emptyset$ (which is in $\mathcal{K}$ by (HP) and $\mathcal{K} \neq \emptyset$). Suppose $\mathcal{D}_i$ has been constructed. By (JEP) there is $\mathcal{D}'_{i+1} \in \mathcal{K}$, such that $\mathcal{D}_i$ and $\mathcal{A}_i$ embed in $\mathcal{D}'_{i+1}$. Let (by Lemma 2.2) $\mathcal{D}_{i+1} \cong \mathcal{D}'_{i+1}$ with $\mathcal{D}_i \leq \mathcal{D}_{i+1}$. Then also $\mathcal{A}_i$ embeds in $\mathcal{D}_{i+1}$.

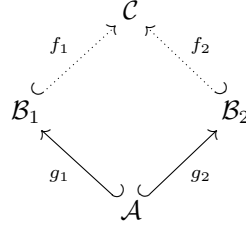
Let $\mathcal{M} := \bigcup_{i \in \omega} \mathcal{D}_i$, which is countable since each $\mathcal{D}_i$ is finite. Then each $\mathcal{A}_i$ embeds in $\mathcal{M}$, so $\mathcal{K} \subseteq \text{age}(\mathcal{M})$. Conversely, if $\mathcal{A} \leq \mathcal{M}$, then $\mathcal{A} \leq \mathcal{D}_i$ for some i , so $\mathcal{A} \in \text{age}(\mathcal{K})$ by (HP). $\square$

Lemma 8.36. *Any consistent $\mathcal{L}$ -theory with QE and with infinite models is $\aleph_0$ -categorical.*

Proof. For any $n \in \omega$ there are only finitely many atomic $\mathcal{L}$ -formulas in free variables $x_1, \dots, x_n$, and hence only finitely many qf $\mathcal{L}$ -formulas in $\bar{x}$ up to equivalence. So we conclude by Ryll-Nardzewski. $\square$

Definition 8.37. A **Fraïssé class** is a class $\mathcal{K}$ of finite $\mathcal{L}$ -structures which contains unboundedly large structures and satisfies (HP) and

(AP) If $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{K}$ and $g_i : \mathcal{A} \hookrightarrow \mathcal{B}_i$ are embeddings, then there exist $\mathcal{C} \in \mathcal{K}$ and embeddings $f_i : \mathcal{B}_i \hookrightarrow \mathcal{C}$, such that $f_1 \circ g_1 = f_2 \circ g_2$.



Remark 8.38. (AP) implies (JEP): take $\mathcal{A} := \emptyset$.

Theorem 8.39 (Fraïssé).

(i) *Let $\mathcal{K}$ be a Fraïssé class. Then there is a unique $\mathcal{L}$ -theory $T_{\mathcal{K}}$, such that $T_{\mathcal{K}}$ has QE and infinite models and $\mathcal{K}$ is the age of any model of $T_{\mathcal{K}}$.*

*The unique countable model of $T_{\mathcal{K}}$ is called the **Fraïssé limit** of $\mathcal{K}$.*

(ii) *Conversely, if $\mathcal{M}$ is an infinite $\mathcal{L}$ -structure with QE, then $\text{age}(\mathcal{M})$ is a Fraïssé class.*

Example 8.40.

- The class of finite linear orders is a Fraïssé class with Fraïssé limit.
- The class of finite graphs is a Fraïssé class with Fraïssé limit the countable random graph.

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Proof. For $\mathcal{A}$ a finite $\mathcal{L}$ -structure: say $\mathcal{A} = \{a_1, \dots, a_n\}$, let

$$\phi_{\mathcal{A}, \bar{a}}(\bar{x}) := \bigwedge \{ \phi(\bar{x}) : \phi(\bar{x}) \text{ basic; } \mathcal{A} \models \phi(\bar{a}) \}.$$

So for $\mathcal{M}$ an $\mathcal{L}$ -structure and $\bar{a}' \in \mathcal{M}^n$, we have $\mathcal{M} \models \phi_{\mathcal{A}, \bar{a}}(\bar{a}')$ iff $\bar{a}' \mapsto \bar{a}$ defines an isomorphism $\langle \bar{a}' \rangle^{\mathcal{M}} \xrightarrow{\cong} \mathcal{A}$.

(ii) Suppose $\mathcal{M}$ is an infinite $\mathcal{L}$ -structure with QE. Set $\mathcal{K} := \text{age}(\mathcal{M})$.

Claim. Suppose $g : \mathcal{A} \hookrightarrow \mathcal{B} \in \mathcal{K}$ and $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{B} = \{g(a_1), \dots, g(a_n), b_1, \dots, b_m\}$. Then

$$\mathcal{M} \models \theta_{\mathcal{A}, \mathcal{B}, g, \bar{a}, \bar{b}} := \forall \bar{x}. (\phi_{\mathcal{A}, \bar{a}}(\bar{x}) \rightarrow \exists \bar{y}. \phi_{\mathcal{B}, g(\bar{a})\bar{b}}(\bar{x}, \bar{y})).$$

Proof. $\mathcal{B} \in \mathcal{K} = \text{age}(\mathcal{M})$, so let $\bar{c}\bar{d} \in \mathcal{M}^{<\omega}$ with $\mathcal{M} \models \phi_{\mathcal{B}, g(\bar{a})\bar{b}}(\bar{c}, \bar{d})$. Then $\mathcal{M} \models \exists \bar{y}. \phi_{\mathcal{B}, g(\bar{a})\bar{b}}(\bar{c}, \bar{y})$. Now $\mathcal{M} \models \phi_{\mathcal{A}, \bar{a}}(\bar{c})$, and by QE $\exists \bar{y}. \phi_{\mathcal{B}, g(\bar{a})\bar{b}}(\bar{x}, \bar{y})$ is equivalent to a qf formula, so it is implied by $\phi_{\mathcal{A}, \bar{a}}(\bar{x})$. $\square$

We show that $\mathcal{K}$ is a Fraïssé class.

(HP) holds by Lemma 8.34.

Let $g_i : \mathcal{A} \hookrightarrow \mathcal{B}_i$ be as in (AP). Composing with an isomorphism, we may assume $\mathcal{A} \leq \mathcal{M}$.

Say $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{B}_i = \{g_i(a_1), \dots, g_i(a_n), b_1^i, \dots, b_{m_i}^i\}$.

Then by the Claim, $\mathcal{M} \models \theta_{\mathcal{A}, \mathcal{B}_i, g_i, \bar{a}, \bar{b}^i}$,

so $\mathcal{M} \models \bigwedge_{i=1,2} \exists \bar{y}. \phi_{\mathcal{B}_i, g_i(\bar{a})\bar{b}^i}(\bar{a}, \bar{y})$.

Let $\bar{c}^1$ and $\bar{c}^2$ be witnesses; then

$$f_i(g_i(\bar{a})) := \bar{a}; \quad f_i(\bar{b}^i) := \bar{c}^i$$

defines embeddings $f_i : \mathcal{B}_i \hookrightarrow \mathcal{M}$ with $f_1 \circ g_1 = f_2 \circ g_2$.

So we obtain (AP) by setting $\mathcal{C} := \langle f_1(\mathcal{B}_1) \cup f_2(\mathcal{B}_2) \rangle^{\mathcal{M}}$.

(i) Suppose $\mathcal{K}$ is a Fraïssé class.

For $n \in \omega$, let $\mathcal{K}_n \subseteq_{\text{fin}} \mathcal{K}$ be such that any $\mathcal{A} \in \mathcal{K}$ with $|\mathcal{A}| \leq n$ is isomorphic to some $\mathcal{A}' \in \mathcal{K}_n$, and let

$$\chi_n(x_1, \dots, x_n) := \forall \bar{x}. \bigvee \{ \phi_{\mathcal{A}, \bar{a}}(\bar{x}) : \mathcal{A} \in \mathcal{K}_n; \mathcal{A} = \{a_1, \dots, a_n\} \}.$$

Let $\Theta_{\mathcal{K}}$ be the class of triples $(\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b})$ such that $\mathcal{A} \leq \mathcal{B} \in \mathcal{K}$ and $\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{B} = \{a_1, \dots, a_n, b_1, \dots, b_m\}$.

Let $\theta_{\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b}} := \theta_{\mathcal{A}, \mathcal{B}, \text{id}_{\mathcal{A}}, \bar{a}, \bar{b}}$.

Let

$$T_{\mathcal{K}} := \{ \theta_{\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b}} : (\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b}) \in \Theta_{\mathcal{K}} \} \cup \{ \chi_n : n \in \omega \}.$$

Any model of $T_{\mathcal{K}}$ has age $\mathcal{K}$, since it satisfies the and $\theta_{\emptyset, \mathcal{B}, \emptyset, \bar{b}}$. By the above Claim, if $\mathcal{M}$ has QE and $\text{age}(\mathcal{M}) = \mathcal{K}$, then $\mathcal{M} \models T_{\mathcal{K}}$.

It remains to see that $T_{\mathcal{K}}$ has QE and infinite models; indeed, it is then complete (by Corollary 5.16), and the claimed uniqueness follows.

We verify QE via Theorem 5.15(iii).

Suppose $\mathcal{A} \leq \mathcal{M}_1, \mathcal{M}_2 \models T_{\mathcal{K}}$ is a finite common substructure of models of $T_{\mathcal{K}}$.

Say $\mathcal{A} = \{a_1, \dots, a_n\}$.

Suppose $\exists y. \psi(\bar{x}, y)$ is a primitive existential $\mathcal{L}$ -formula

and $\mathcal{M}_1 \models \psi(\bar{a}, b)$ for some $b \in \mathcal{M}_1$.

Let $\mathcal{B} = \langle \bar{a}b \rangle^{\mathcal{M}_1}$.

Since $\mathcal{M}_1 \models \chi_{n+1}$, we have $\mathcal{A}, \mathcal{B} \in \mathcal{K}$.

Then $\psi(\bar{x}, y)$ is implied by $\phi_{\mathcal{B}, \bar{a}b}(\bar{x}, y)$.

So since $\mathcal{M}_2 \models \theta_{\mathcal{A}, \mathcal{B}, \bar{a}, b}$ and $\mathcal{M}_2 \models \phi_{\mathcal{A}, \bar{a}}(\bar{a})$,

also $\mathcal{M}_2 \models \exists y. \psi(\bar{a}, y)$, as required.

Finally, we construct an infinite model of $T_{\mathcal{K}}$.

Claim. *Let $(\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b}) \in \Theta$, and suppose $\mathcal{A} \leq \mathcal{D} \in \mathcal{K}$. Then there exists $\mathcal{D}' \in \mathcal{K}$ such that $\mathcal{D} \leq \mathcal{D}'$, and there is an embedding $f : \mathcal{B} \hookrightarrow \mathcal{D}'$ such that $f|_{\mathcal{A}} = \text{id}|_{\mathcal{A}}$.*

Proof. Immediate consequence of (AP) (and Lemma 2.2). $\square$

Let $\xi : \omega \times \omega \rightarrow \omega$ be a bijection such that $\xi(i, j) \geq i$.

We construct a countable chain $\mathcal{D}_0 \leq \mathcal{D}_1 \leq \dots$ with $\mathcal{D}_k \in \mathcal{K}$.

We simultaneously construct a sequence $(\mathcal{A}_i, \bar{a}_i)_{i \in \omega}$,

and for $k \in \omega$ some $m_k > k$,

such that if $\bar{a}$ is a tuple of distinct elements of $\mathcal{D}_k$,

then $\bar{a} = \bar{a}_i$ for some $i < m_k$

and for all $i < m_k$, we have $\mathcal{A}_i = \langle \bar{a}_i \rangle^{\mathcal{D}_k} \leq \mathcal{D}_k$.

For each $i \in \omega$ we take (using Remark 8.32) a sequence $(\mathcal{B}_j^i, \bar{b}_j^i)_{j \in \omega}$ with

$(\mathcal{A}_i, \mathcal{B}_j^i, \bar{a}_i, \bar{b}_j^i) \in \Theta$,

such that if $(\mathcal{A}_i, \mathcal{B}, \bar{a}_i, \bar{b}) \in \Theta$, then

$$\bar{b} \mapsto \bar{b}_j^i; \bar{a}_i \mapsto \bar{a}_i$$

defines an isomorphism $\mathcal{B} \xrightarrow{\cong} \mathcal{B}_j^i$.

Let $\mathcal{D}_0 := \emptyset$ (and $\mathcal{A}_0 := \emptyset$ and $m_0 := 1$).

Given $\mathcal{D}_k$, say $k = \xi(i, j)$,

we have $i \leq \xi(i, j) = k < m_k$, so $\mathcal{A}_i \leq \mathcal{D}_k$.

By the Claim we find $\mathcal{D}_k \leq \mathcal{D}_{k+1} \in \mathcal{K}$,

such that $\mathcal{B}_j^i$ embeds in $\mathcal{D}_{k+1}$ over $\mathcal{A}_i$;

i.e. $\mathcal{D}_{k+1} \models \exists y. \phi_{\mathcal{B}_j^i, \bar{a}_i \bar{b}}(\bar{a}_i, y)$.

We may extend $(\mathcal{A}_i, \bar{a}_i)_i$ to include the substructures of $\mathcal{D}_{k+1}$, and set $m_{k+1} > m_k$ correspondingly.

So then $\mathcal{M}_{\mathcal{K}} := \bigcup_{k \in \omega} \mathcal{D}_k \models \theta_{\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b}}$ for all $(\mathcal{A}, \mathcal{B}, \bar{a}, \bar{b}) \in \Theta$.

Also $\mathcal{M} \models \chi_n$, since $\mathcal{D}_k \in \mathcal{K}$. So $\mathcal{M} \models T_{\mathcal{K}}$.

Finally, $\mathcal{M}$ is infinite, since $\mathcal{K}$ contains unboundedly large finite structures.

□

Remark 8.41. Analogues of Theorem 8.39 exist for arbitrary countable languages, with finitely generated substructures in place of finite substructures.

In non-relational languages, one must assume (JEP) as well as (AP), or equivalently assume (AP) and: $\langle \emptyset \rangle^{\mathcal{A}} \cong \langle \emptyset \rangle^{\mathcal{B}}$ for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}$.

In general $\aleph_0$ -categoricity and QE are weakened to *ultrahomogeneity* of the Fraïssé limit: any partial automorphism with f.g. domain extends to an automorphism. The proof is essentially the same.

There are further generalisations, including “Hrushovski-Fraïssé constructions”, in which (HP) is relaxed.

End of lecture 15

9 Cofinality and regularity

Definition 9.1. Let X be a linearly ordered set.

- A subset $A \subseteq X$ is **cofinal** if $\forall b \in X. \exists a \in A. a \geq b$.
- The **cofinality** of X , written $\text{cof}(X)$, is the minimal cardinality of a cofinal subset.
- An infinite cardinal λ is **regular** if $\text{cof}(\lambda) = \lambda$, otherwise it is **singular**.

Lemma 9.2. Suppose λ is a regular cardinal and $(A_\alpha)_{\alpha \in \lambda}$ is an increasing chain of sets. Suppose $B \subseteq \bigcup_{\alpha \in \lambda} A_\alpha$ with $|B| < \lambda$. Then there exists an ordinal $\alpha \in \lambda$ such that $B \subseteq A_\alpha$.

Proof. Otherwise, $\{\inf\{\alpha : b \in A_\alpha\} : b \in B\}$ is cofinal, so $\text{cof}(\lambda) \leq |B| < \lambda$, contradicting regularity. □

Proposition 9.3. Infinite successor cardinals are regular.

Proof. Let κ be an infinite cardinal. Suppose $A \subseteq \kappa^+$ with $|A| < \kappa^+$. Then $|A| \leq \kappa$ and every $\alpha \in A$ has cardinality $|\alpha| \leq \kappa$, so $|\bigcup_{\alpha \in A} \alpha| \leq \kappa \cdot \kappa = \kappa$. But then $\sup_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha < \kappa^+$, so A is not cofinal in κ^+ .

So $\text{cof}(\kappa^+) = \kappa^+$. □

10 Saturation

10.1 Existence

Definition 10.1. For $A \subseteq B \subseteq \mathcal{M}$, define

$$\restriction_A : S(B) \rightarrow S(A); \text{tp}(a/B) \mapsto \text{tp}(a/A)$$

(for $a \in \mathcal{M}' \succeq \mathcal{M}$).

Proposition 10.2. *Let T be a theory with infinite models.*

Let $\kappa \leq \lambda \geq |T|$ be infinite cardinals such that for any $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \lambda$, there exists a subset $\Xi_{\mathcal{M}} \subseteq S_1(\mathcal{M})$ such that for any $A \subseteq \mathcal{M}$ with $|A| \leq \kappa$ we have $|_A(\Xi_{\mathcal{M}}) = S_1(A)$.

Then T has a κ^+ -saturated model of cardinality $\lambda + \kappa^+$.

Corollary 10.3. *Let T be a theory with infinite models.*

For any infinite cardinal $\kappa \geq |T|$, T has a κ^+ -saturated model of cardinality 2^κ .

In particular, T has a μ -saturated model for any infinite cardinal μ .

Proof. If $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \lambda$ und $A \subseteq \mathcal{M}$ with $|A| \leq \kappa$, then $|S_1(A)| \leq 2^{|T_A|} = 2^{|T|+|A|} \leq 2^\kappa$. Meanwhile

$$|A \subseteq \mathcal{M} : |A| \leq \kappa| \leq |\mathcal{M}|^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa.$$

Then

$$\left| \bigcup_{A \subseteq \mathcal{M}; |A| \leq \kappa} S_1(A) \right| \leq 2^\kappa \cdot 2^\kappa = 2^\kappa = \lambda.$$

So we can find $|\Xi_{\mathcal{M}}| \leq \lambda$ as required.

Finally, $2^\kappa + \kappa^+ = 2^\kappa$. □

Bemerkung 10.4. It follows that if $\lambda^+ = 2^\lambda$, e.g. if we assume GCH, then T has a saturated model of cardinality λ^+ . But (under a “reasonable” large cardinal assumption) it is consistent with ZFC that no infinite λ exists with $2^\lambda = \lambda^+$.

Proof of Proposition 10.2. We build an elementary chain $(\mathcal{M}_\alpha)_{\alpha \in \kappa^+}$ of models of cardinality λ .

Let (by Löwenheim-Skolem) $\mathcal{M}_0 \models T$ with $|\mathcal{M}_0| = \lambda$. For $\eta \in \kappa^+$ a limit ordinal, set $\mathcal{M}_\eta := \bigcup_{\alpha < \eta} \mathcal{M}_\alpha$. By Lemma 8.1, $\mathcal{M}_\eta \succeq \mathcal{M}_\alpha$ for $\alpha < \eta$. Since $|\eta| \leq \kappa \leq \lambda$, we have $|\mathcal{M}_\eta| = \lambda$.

Given $\mathcal{M}_\alpha$, By Lemma 7.16 and Löwenheim-Skolem there is $\mathcal{M}_{\alpha+1} \succeq \mathcal{M}_\alpha$ with $|\mathcal{M}_{\alpha+1}| = \lambda$ which realises every type in $\Xi_{\mathcal{M}_\alpha}$, and hence every type in $S_1(A)$ for $A \subseteq \mathcal{M}_\alpha$ with $|A| \leq \kappa$.

Now let $\mathcal{M} := \bigcup_{\alpha \in \kappa^+} \mathcal{M}_\alpha$. Then $\lambda \leq |\mathcal{M}| \leq \lambda \cdot \kappa^+ = \lambda + \kappa^+$.

If $A \subseteq \mathcal{M}$ with $|A| < \kappa^+$, then since κ^+ is a successor cardinal and hence is regular, already $A \subseteq \mathcal{M}_\alpha$ for some $\alpha \in \kappa^+$. So any $p \in S(A)$ is realised in $\mathcal{M}_{\alpha+1} \preceq \mathcal{M}$ and hence in $\mathcal{M}$. Hence $\mathcal{M}$ is κ^+ -saturated.

It follows that $|\mathcal{M}| \geq \kappa^+$, so $|\mathcal{M}| = \lambda + \kappa^+$. □

10.2 Stability

Definition 10.5. Let κ be an infinite cardinal.

A structure $\mathcal{M}$ is κ -**stable** if $|S_1(A)| \leq \kappa$ for all $A \subseteq \mathcal{M}$ with $|A| \leq \kappa$.

A theory T is κ -**stable** if it has infinite models and every $\mathcal{M} \models T$ is κ -stable.

A model or theory is **stable** if it is κ -stable for some infinite cardinal κ .

We often write ω -stable for $\aleph_0$ -stable.

Example 10.6. T_∞ and $T_{(\mathbb{Q}; +)}$ are ω -stable.

DLO is not stable.

Corollary 10.7 (of Proposition 10.2). *Let T be λ -stable, where $\lambda \geq |T|$.*

- (i) *T has a saturated model of cardinality λ^+ .*
- (ii) *Let $\kappa < \lambda$. Then T has a κ^+ -saturated model of cardinality λ .*

Proof. Let $\kappa \leq \lambda$. By Proposition 10.2, with $\Xi_{\mathcal{M}} := S_1(\mathcal{M})$, there is a κ^+ -saturated model of cardinality $\lambda + \kappa^+$.

Then (i) follows by taking $\kappa := \lambda$, and (ii) with $\kappa < \lambda$. $\square$

End of lecture 16

10.3 QE and saturation

Notation 10.8.

- Let $\mathcal{M}$ an $\mathcal{L}$ -structure. For $\bar{b} \in \mathcal{M}^{<\omega}$,

$$\text{qftp}^{\mathcal{M}}(\bar{b}/A) := \{\phi(\bar{x}) : \phi \text{ qf } \mathcal{L}(A)\text{-formula; } \mathcal{M} \models \phi(\bar{b})\}.$$

- Let $\mathcal{M}_1, \mathcal{M}_2$ be $\mathcal{L}$ -structures. For $\bar{b}_i \in \mathcal{M}_i^{<\omega}$ we write

$$\bar{b}_1 \equiv \bar{b}_2$$

when $\text{tp}^{\mathcal{M}_1}(\bar{b}_1) = \text{tp}^{\mathcal{M}_2}(\bar{b}_2)$, and we write

$$\bar{b}_1 \equiv^{\text{qf}} \bar{b}_2$$

when $\text{qftp}^{\mathcal{M}_1}(\bar{b}_1) = \text{qftp}^{\mathcal{M}_2}(\bar{b}_2)$.

- If $A \subseteq \mathcal{M}_1, \mathcal{M}_2$ is a common subset, we write

$$\bar{b}_1 \equiv_A \bar{b}_2$$

when $\text{tp}^{\mathcal{M}_1}(\bar{b}_1/A) = \text{tp}^{\mathcal{M}_2}(\bar{b}_2/A)$; ähnlich für $\bar{b}_1 \equiv_A^{\text{qf}} \bar{b}_2$.

Remark 10.9.

- $\bar{b}_1 \equiv \bar{b}_2$ iff $\bar{b}_1 \mapsto \bar{b}_2$ defines a partial elementary map $\mathcal{M}_1 \dashrightarrow \mathcal{M}_2$.
- $\bar{b}_1 \equiv^{\text{qf}} \bar{b}_2$ iff $\bar{b}_1 \mapsto \bar{b}_2$ defines a partial isomorphism $\mathcal{M}_1 \dashrightarrow \mathcal{M}_2$.

Proposition 10.10. *Let T be a theory. Let κ be an infinite cardinal. TFAE:*

- (i) *T has QE;*
- (ii) *Let $\mathcal{M}_1, \mathcal{M}_2 \models T$ be κ -saturated models, $\bar{a}_i \in \mathcal{M}_i^{<\omega}$ with $\bar{a}_1 \equiv^{\text{qf}} \bar{a}_2$, and $b_1 \in \mathcal{M}_1$.*

Then there exists $b_2 \in \mathcal{M}_2$ with $\bar{a}_1 b_1 \equiv^{\text{qf}} \bar{a}_2 b_2$.

Proof.

- (i) $\Rightarrow$ (ii) Let $\theta : \bar{a}_1 \mapsto \bar{a}_2$. This is a partial isomorphism, so by QE it is partial elementary. Hence (by Lemma 7.20) $\text{tp}(b_1/\bar{a}_1)^\theta$ a type in $S^{\mathcal{M}_2}(\bar{a}_2)$. So by $\aleph_0$ -saturation of $\mathcal{M}_2$, it is realised in $\mathcal{M}_2$, say by $b_2 \in \mathcal{M}_2$. Then $\bar{a}_1 b_1 \equiv \bar{a}_2 b_2$.

(ii) $\Rightarrow$ (i) We verify QE via Theorem 5.15(iii).

Let $\mathcal{A} = \langle \bar{a} \rangle \leq \mathcal{M}_1, \mathcal{M}_2 \models T$ be a common finitely generated substructure of two models of T , and $\exists y. \psi(\bar{x}, y)$ a primitive existential formula, and $b_1 \in \mathcal{M}_1$ with $\mathcal{M}_1 \models \psi(\bar{a}, b_1)$.

By Corollary 10.3, we can find κ -saturated elementary extensions $\mathcal{M}'_i \succeq \mathcal{M}_i$. By (ii), there is $b_2 \in \mathcal{M}'_2$ with $\bar{a}b_1 \equiv^{\text{qf}} \bar{a}b_2$, hence $\mathcal{M}_2 \models \exists y. \psi(\bar{a}, y)$, as required.

□

Example 10.11. An *ordered $\mathbb{Q}$ -vector space* is a $\mathbb{Q}$ -vector space with a linear ordering $<$ such that

$$\forall x, y, z. (x < y \rightarrow x + z < y + z).$$

Let $T_{\text{g}\mathbb{Q}\text{-VR}}$ be the $\{0, +, (q \cdot)_{q \in \mathbb{Q}}, <\}$ -theory consisting of this axiom along with axioms for non-trivial $\mathbb{Q}$ -vector spaces.

Claim. $T_{\text{g}\mathbb{Q}\text{-VR}}$ has QE and is complete.

Proof. Completeness follows from QE via Corollary 5.16, since $\langle \emptyset \rangle^{\mathcal{M}} = \{0\}$ for any $\mathcal{M} \models T_{\text{g}\mathbb{Q}\text{-VR}}$.

Let $\mathcal{M}_1, \mathcal{M}_2 \models T_{\text{g}\mathbb{Q}\text{-VR}}$ be $\aleph_1$ -saturated. Let $\bar{a}_i \in \mathcal{M}_i^{<\omega}$ with $\bar{a}_1 \equiv^{\text{qf}} \bar{a}_2$. Let $b_1 \in \mathcal{M}_1$.

A qf-type $\text{qftp}(b/\bar{a})$ is determined by the formulas of form $x = \sum q_i \cdot a_i$ or $x < \sum q_i \cdot a_i$ or $x > \sum q_i \cdot a_i$.

Let $\mathcal{A}_i := \langle \bar{a}_i \rangle^{\mathcal{M}_i}$. Then $\bar{a}_1 \mapsto \bar{a}_2$ generates an order isomorphism $\theta : \mathcal{A}_1 \rightarrow \mathcal{A}_2$. Note that the reduct $\mathcal{M}_i \upharpoonright_{<}$ of $\mathcal{M}_i$ to an order is a model of DLO. So by QE for DLO, θ is a partial elementary map of linear orders $\theta : \mathcal{M}_1 \upharpoonright_{<} \dashrightarrow \mathcal{M}_2 \upharpoonright_{<}$, so $(\text{tp}^{\mathcal{M}_1 \upharpoonright_{<}}(b_1/\mathcal{A}_1))^\theta$ is a type in $S_1^{\mathcal{M}_2 \upharpoonright_{<}}(\mathcal{A}_2)$. By $\aleph_1$ -saturation of $\mathcal{M}_2$, say $b_2 \in \mathcal{M}_2$ realises this type.

Then $\bar{a}_1 b_1 \equiv^{\text{qf}} \bar{a}_2 b_2$. □

One can deduce that the $\{0, +, <\}$ -theory DOAG of linearly ordered divisible abelian groups (e.g. $(\mathbb{Q}; 0, +, <)$) is complete and has QE.

10.4 Bonus: Monsters

Definition 10.12. A structure $\mathcal{M}$ is **strongly κ -homogeneous**, if any partial elementary map $\theta : \mathcal{M} \dashrightarrow \mathcal{M}$ with $|\text{dom } \theta| < \kappa$ extends to an automorphism of $\mathcal{M}$.

$\mathcal{M}$ is **strongly κ -saturated** if it is κ -saturated and strongly κ -homogeneous.

Proposition 10.13. *If a structure $\mathcal{M}$ is saturated, it is strongly $|\mathcal{M}|$ -saturated.*

Proof. As in ÜA 7.2(a).

Briefly: If $\theta : \mathcal{M} \dashrightarrow \mathcal{M}$ is partial elementary and $|\text{dom } \theta| < |\mathcal{M}|$, then θ induces an elementary equivalence between the expansions by constants $\mathcal{M}_{\text{dom } \theta} \equiv \mathcal{M}_{\text{im } \theta}$; but these structures are also saturated, so by Lemma 7.21 they are isomorphic. An isomorphism between these structures is an automorphism of $\mathcal{M}$ extending θ . □

Theorem 10.14. *Let T be a theory with infinite models. Let λ be an infinite cardinal. Then T has a strongly λ -saturated model.*

Proof. Increasing λ , we may assume $\lambda \geq |T|$ and λ is regular.

Let $\mathcal{M}_0 \models T$ with $|\mathcal{M}_0| > \lambda$. By Corollary 10.3 we may extend to an elementary chain $(\mathcal{M}_\alpha)_{\alpha \in \lambda}$ such that each $\mathcal{M}_{\alpha+1}$ is $|\mathcal{M}_\alpha|$ -saturated, and for η a limit ordinal $\mathcal{M}_\eta = \bigcup_{\alpha < \eta} \mathcal{M}_\alpha$.

Let $\mathcal{M} := \bigcup_{\alpha \in \lambda} \mathcal{M}_\alpha$. Then $\mathcal{M}$ is λ -saturated since each $\mathcal{M}_{\alpha+1}$ is and λ is regular.

Now let $\theta : \mathcal{M} \dashrightarrow \mathcal{M}$ be partial elementary with $|\text{dom } \theta| < \lambda$. Since λ is regular, for some $\alpha \in \lambda$ we have $\theta : \mathcal{M}_\alpha \dashrightarrow \mathcal{M}_\alpha$, i.e. $\text{dom } \theta \cup \text{im } \theta \subseteq \mathcal{M}_\alpha$. Let $\theta_\alpha := \theta$.

Now $\mathcal{M}_{\alpha+1}$ is $|\mathcal{M}_\alpha|$ -saturated, so as in the proof of Lemma 7.23, θ_α extends to a partial elementary map $\theta_{\alpha+1} : \mathcal{M}_{\alpha+1} \dashrightarrow \mathcal{M}_{\alpha+1}$ with $\mathcal{M}_\alpha \subseteq \text{dom } \theta_{\alpha+1}$ and also $\mathcal{M}_\alpha \subseteq \text{im } \theta_{\alpha+1}$.

We obtain in this way, taking unions at limit ordinals, an increasing chain $(\theta_\beta)_{\alpha \leq \beta \in \lambda}$ of partial elementary maps $\theta_\beta : \mathcal{M}_\beta \dashrightarrow \mathcal{M}_\beta$ with each $\mathcal{M}_\beta \subseteq \text{dom } \theta_{\beta+1} \cap \text{im } \theta_{\beta+1}$.

Then $\sigma := \bigcup_\beta \theta_\beta \in \text{Aut}(\mathcal{M})$ as required. $\square$

11 ω -stability

Let T be a complete $\mathcal{L}$ -theory with infinite models.

Lemma 11.1. *T is κ -stable iff for all $A \subseteq \mathcal{M} \models T$ with $|A| \leq \kappa$, we have $|S(A)| \leq \kappa$.*

In particular, T is ω -stable iff $\text{Th}(\mathcal{M}_A)$ is small for all $A \subseteq \mathcal{M} \models T$ with $|A| = \aleph_0$.

Proof. We show by induction that $|S_n(A)| \leq \kappa$ for all n . For $n = 1$, this is the definition of κ -stability.

Suppose $|S_n(A)| \leq \kappa$.

Let $\mathcal{N} \succeq \mathcal{M}$ be κ^+ -saturated.

Let $(\bar{b}_i \in \mathcal{N}^n)_{i \in \kappa}$ be such that $\{\text{tp}(\bar{b}_i/A) : i \in \kappa\} = S_n(A)$.

Let $\bar{b}c \in \mathcal{N}^{n+1}$.

Then $\bar{b}c \equiv_A \bar{b}_i c'$ for some $i \in \kappa$ and $c' \in \mathcal{N}$.

For each $i \in \kappa$ there are at most $|S_1(A \cup \bar{b}_i)| \leq \kappa$ possibilities for $\text{tp}(c'/\bar{b}_i)$.

So $|S_{n+1}| \leq \kappa \cdot \kappa = \kappa$. $\square$

End of lecture 17

Definition 11.2. T is **totally transcendental** if for any $\mathcal{M} \models T$, there is no binary tree $(\phi_s)_{s \in 2^{<\omega}}$ of formulas for $\text{Th}(\mathcal{M}_\mathcal{M})$.

Theorem 11.3.

(a) *If T is ω -stable then T is totally transcendental.*

(b) *If T is totally transcendental then T is κ -stable for all $\kappa \geq |T|$.*

Corollary 11.4. *Suppose $|T| = \aleph_0$. TFAE:*

- (i) T is ω -stable;
- (ii) T is totally transcendental.
- (iii) T is κ -stable for all $\kappa \geq \aleph_0$.

Proof of Theorem 11.3.

- (a) Suppose $\mathcal{M} \models T$ and $\text{Th}(\mathcal{M}_{\mathcal{M}})$ has a binary tree of formulas $(\phi_s)_{s \in 2^{<\omega}}$. Since $|2^{<\omega}| = \aleph_0$, there exist $A \subseteq \mathcal{M}$ with $|A| \leq \aleph_0$ such that each ϕ_s is a $\mathcal{L}(A)$ -formula, and so $(\phi_s)_s$ is also a binary tree of formulas for $\text{Th}(\mathcal{M}_A)$. Then by Lemma 8.20, $|S^{\mathcal{M}}(A)| = |S(\text{Th}(\mathcal{M}_A))| = 2^{\aleph_0} > \aleph_0$, so T is not ω -stable.

- (b) (c.f. $\ddot{\text{U}}\text{A}$ 6.1(a))

Let $A \subseteq \mathcal{M} \models T$ with $|A| \leq \kappa$. For $\phi(x)$ an $\mathcal{L}(A)$ -formula, define $[\phi(x)] := \{p \in S_1(A) : \phi(x) \in p(x)\}$.

Claim. *If $|[\phi(x)]| > \kappa$, then there is an $\mathcal{L}(A)$ -formula $\psi(x)$, such that $|[\phi(x) \wedge \psi(x)]| > \kappa < |[\phi(x) \wedge \neg\psi(x)]|$.*

Proof. Say $p \in S_1(A)$ is *small* if $\exists \psi \in p. |\psi| \leq \kappa$. There are at most $|\mathcal{L}(A)| \cdot \kappa \leq \kappa \cdot \kappa = \kappa$ small types in $S_1(A)$. Hence there are distinct non-small types $p_1, p_2 \in [\phi(x)]$. Let $\psi(x) \in p_1(x) \setminus p_2(x)$; then $\phi(x) \wedge \psi(x) \in p_1(x)$ and $\phi(x) \wedge \neg\psi(x) \in p_2(x)$, so ψ is as required. $\square$

Now if $|[x \dot{=} x]| = |S_1(A)| > \kappa$, we build a binary tree with $|[\phi_s]| > \kappa$ for all $s \in 2^{<\omega}$: set $\phi_\emptyset := x \dot{=} x$; given ϕ_s , let $\phi_{s0} := \phi_s \wedge \psi$ and $\phi_{s1} := \phi_s \wedge \neg\psi$, where ψ is as in the claim. $\square$

11.1 Constructibility

Notation 11.5. If $A \subseteq \mathcal{M} \models T$ and $\mathcal{N} \models T$, we call a map $\theta : A \rightarrow \mathcal{N}$ *partial elementary (p.e.)* and write $\theta : A \xrightarrow{\equiv} \mathcal{N}$, if the corresponding partial map $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ is p.e..

Remark 11.6. $\theta : A \rightarrow \mathcal{N} \models T$ is p.e. iff $\mathcal{N}_{A,\theta} \equiv \mathcal{M}_A$, where $\mathcal{N}_{A,\theta}$ is the $\mathcal{L}(A)$ -structure defined by $a^{\mathcal{N}_{A,\theta}} := \theta(a)$.

Definition 11.7. Let $A \subseteq \mathcal{M} \models T$. Then $\mathcal{M}$ is **prime over** A if every p.e. map $A \xrightarrow{\equiv} \mathcal{N} \models T$ extends to an elementary embedding $\mathcal{M}(\xrightarrow{\equiv}) \mathcal{N}$.

Remark 11.8. Let $A \subseteq \mathcal{M} \models T$. Applying Remark 11.6, we find that $\mathcal{M}$ is prime over A iff $\mathcal{M}_A$ is a prime model of $\text{Th}(\mathcal{M}_A)$.

Remark 11.9. By Proposition 8.23 and Proposition 8.15, a countable ω -stable theory has a unique prime model over any *countable* set $A \subseteq \mathcal{M} \models T$.

In fact this holds also for uncountable A . We prove existence in this section. Uniqueness requires further work.

Definition 11.10. If $A \subseteq B \subseteq \mathcal{M} \models T$, then B is **constructible** over A if B can be enumerated as $(b_\alpha)_{\alpha < \gamma}$ for some ordinal γ , such that for all $\alpha < \gamma$ the type $\text{tp}(b_\alpha/A \cup b_{<\alpha})$ is isolated, where $b_{<\alpha} := \{b_\beta : \beta < \alpha\}$.

Lemma 11.11. *If $\mathcal{M} \models T$ is constructible over $A \subseteq \mathcal{M}$, then $\mathcal{M}$ is prime over A .*

Proof. Suppose $\theta_0 : A \xrightarrow{\equiv} \mathcal{N} \models T$. We construct a chain of p.e. maps $\theta_\alpha : A \cup b_{<\alpha} \xrightarrow{\equiv} \mathcal{N} \models T$ for $\alpha \in \gamma$. Then $\bigcup_{\alpha \in \gamma} \theta_\alpha : \mathcal{M}(\xrightarrow{\equiv} \mathcal{N}$ is an elementary embedding.

For $\eta \in \gamma$ a limit ordinal, $\theta_\eta := \bigcup_{\alpha \in \eta} \theta_\alpha$.

Given θ_α , let $\theta_{\alpha+1}(b_\alpha)$ be a realisation in $\mathcal{N}$ of the isolated type $\text{tp}(b_\alpha/A \cup b_{<\alpha})^{\theta_\alpha}$. $\square$

Lemma 11.12. *If T is totally transcendental and $A \subseteq \mathcal{M} \models T$, then the isolated types are dense in $S(A)$.*

Proof. Immediate consequence of Lemma 8.22(i). $\square$

Theorem 11.13. *If T is totally transcendental and $A \subseteq \mathcal{M} \models T$, then T has a constructible prime model over A .*

Proof. A construction sequence over A is a sequence $(b_\alpha)_{\alpha < \gamma}$ with each $\text{tp}(b_\alpha/A \cup b_{<\alpha})$ isolated.

By Zorn, there is a maximal construction sequence $(b_\alpha)_{\alpha < \gamma}$ in $\mathcal{N}$. Let $M := b_{\leq \gamma} \subseteq \mathcal{N}$. We show by the Tarski Test that M is the domain of an elementary substructure $\mathcal{M} \preceq \mathcal{N}$, which is constructible over A and hence by Lemma 11.11 prime over A , as required. .

So let $\phi(x)$ be an $\mathcal{L}(B)$ -formula such that $\mathcal{N} \models \exists y.\phi(x)$. By Lemma 11.12, let $p(x) \in S(B)$ be isolated with $p(x) \vdash \phi(x)$. Let $c \in \mathcal{N}$ realise p . If $c \notin B$ then we could extend the construction sequence by setting $b_\gamma := c$, contradicting maximality. So $c \in B$, and $\mathcal{N} \models \phi(c)$, as required. $\square$