Compressible types and applications

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Existence of compressible ϕ-types

M a structure, $\phi(x; y)$ a formula, $A \subseteq M^y$.

Theorem (A finite: Xi Chen, Yu Cheng, Bo Teng 2016;
A arbitrary: B, Itay Kaplan, Pierre Simon (BKS) 2021)

If $\mathrm{VC}^*(\phi) \leq d < \infty$, *exists* $d2^{d+1}$*-compressible* $p \in S_\phi(A)$.

Remark

For finite A, Hu-Wu-Li-Wang '17 find cd^2-compressible p.
("Quadratic bound on recursive teaching dimension".)

Rounded average

Definition

The **rounded average** of $p_0, \ldots, p_{2m} \in S_\phi(A)$ is

$$\{\psi : |\{i : \psi \in p_i\}| > m\} \in S_\phi(A).$$

Theorem (BKS)

Suppose $d := \mathrm{VC}^*(\phi) < \infty$. *Then exist* $n = n(d)$ *and* $k = k(d)$ *such that any* $p \in S_\phi(A)$ *is the rounded average of* n k*-compressible types.*

Externally definable sets

Definition

The *externally definable subset* of A defined by $p \in S_\phi(A)$ is the set denoted in this talk by

$$\phi(p; A) := \{a \in A : \phi(x; a) \in p\}.$$

More standard notation: $\phi(b; A)$ *where* $M \prec M' \ni b \vDash p$.

Fact (Shelah)

If ϕ *is* **stable**, *its externally definable sets are uniformly definable, i.e.:*
Exists $\theta(w; y)$ *s.t.*
for any $p \in S_\phi(A)$ *with* $A \subseteq M^y$, $|A| > 1$,
exists $d \in A^w$ *s.t.*

$$\phi(p; A) = \theta(d; A).$$

Converse holds if we allow $A \subseteq M'^y$ *for* $M' \succ M$.

Uniform honest definitions

Definition

$\phi(x;y)$ has **uniform honest definitions** if there is $\theta(w;y)$ s.t.
for any $p \in S_\phi(A)$ with $A \subseteq M^y$, $|A| > 1$,
and for any $A_0 \subseteq_{\text{fin}} A$,
exists $d \in A^w$ s.t.

$$\phi(p;A_0) \subseteq \theta(d;A) \subseteq \phi(p;A).$$

Fact (Chernikov-Simon)

If $\text{Th}(M)$ *is NIP,* ϕ *has uniform honest definitions.*

Local uniform honest definitions

Theorem (BKS)

Any NIP ϕ has uniform honest definitions.

Proof idea.

Let $p \in S_\phi(A)$.
Say $p = \mathrm{Av}(p_0, \ldots, p_{n-1})$ where p_i is k-compressible.
Given A_0, say $p_i \vdash \bigwedge_{j<k} \phi(x, d_{ij})^{\epsilon_{ij}} \vdash p_i|_{A_0}$.
Then

$$\theta((d_{ij})_{i<n, j<k}; y) = \mathrm{Maj}_{i<n} \forall x \left(\bigwedge_{j<k} \phi(x; d_{ij})^{\epsilon_{ij}} \to \phi(x; y) \right)$$

satisfies

$$\phi(p; A_0) \subseteq \theta(d; A) \subseteq \phi(p; A).$$

□

Compressible types

Definition

A type $p \in S^x(A)$ is **compressible** if for all $\phi(x;y)$ exists $\zeta(x;z)$ s.t. for all $A_0 \subseteq_{\text{fin}} A$ exists $a \in A^z$ s.t. $\zeta(x;a) \in p$ and $\zeta(M;a) \subseteq p_\phi|_{A_0}(M)$.

Fact

All types are compressible iff T is distal.
O-minimal $\Rightarrow$ distal.

Using existence of compressible ϕ-types:

Theorem (BKS)

A countable theory T is NIP iff compressible types are dense,
i.e. for every $A \subseteq M \models T$, formula $\theta(x;y)$, and $a \in A^y$,
exists compressible $\theta(x;a) \in p \in S^x(A)$.

Transitivity

T countable NIP.

Theorem ("Rescoping" (BKS))

If $\text{tp}(a/B)$ *is compressible and* $C \subseteq B$*,*
then $\text{tp}^B(a/C)$ *is compressible.*

(But $\text{tp}(a/C)$ might not be compressible.)

Definition

If $C \subseteq B \subseteq M \models T$,
B is **compressible over** C if
$\text{tp}(b/C)$ is compressible for any $b \in B^{<\omega}$.

Corollary ("Transitivity of compressibility")

If $C \subseteq B \subseteq A \subseteq M \models T$*,*
and A *is compressible over* B*, and* B *is compressible over* C*,*
then A *is compressible over* C*.*

Existence of compressible models

T countable NIP.

Theorem (BKS)

For any $A \subseteq M \models T$, exists compressible model over A. [1]

Proof.

By density, build model $M_0 = A \cup \{b_i : i < \lambda\}$ with $\mathrm{tp}(b_i / A \cup b_{<i})$ compressible.
Conclude by transitivity of compressibility and transfinite induction. □

[1] i.e. model of $T_A = \mathrm{Diag}^M(A)$ compressible over A.

Model-theoretic applications

T countable NIP.
Let S be a stable $\emptyset$-definable set (i.e. any $S(x) \wedge \phi(x,y)$ is stable).

Theorem (BKS)

Suppose T is unstable and $M \vDash T$ is $\aleph_0$-saturated.
Then exist arbitrarily large $N \succ M$ with $S(N) = S(M)$.

Theorem (BKS)

Any model of $\mathrm{Th}(S^{\mathrm{ind}})$ *is the reduct of a model of* T*.*

Compressibility in valued fields

Theorem (BKS)

Let $K \vDash \mathrm{ACVF}$ with residue field algebraic over $A \subseteq K^{\mathrm{eq}}$.
Then K is compressible over A.

Proof.

(I) A 1-type $\mathrm{tp}(b/A)$ with $A = \mathrm{acl}^{\mathrm{eq}}(A)$ is incompressible iff it extends the residue field, i.e. $\mathrm{dcl}^{\mathrm{eq}}(Ab) \cap k \supsetneq A \cap k$.

(II) So K is compressibly constructed over A by alternating taking acl and taking compressible non-algebraic extensions.

(III) Hence K is compressible by transitivity.

□

Corollary

If $L \leq K \vDash \mathrm{ACVF}$ is a valued field with finite residue field, then L is compressible (in K) over any $A \subseteq L$.

Incidence bound

This recovers the following result, which was originally proven by B and Jean-François Martin by a more direct method with effective bounds.

Corollary

Let L be a valued field with finite residue field, and let $\phi(x; y)$ be a quantifier-free valued field formula.
Then the binary relation $E := \phi(L; L)$ admits a distal cell decomposition in the sense of Chernikov-Galvin-Starchenko, hence satisfies incidence bounds.

In particular:
Let L be a finitely generated extension of a finite field.
Let $E \subseteq L^n \times L^m$ be a $K_{d,s}$-free zero-set of polynomials.
Then there exist $C, \epsilon > 0$ such that for any finite $A \subseteq L^n$ and $B \subseteq L^m$,

$$|E \cap (A \times B)| \leq C(|A|^{1-\epsilon}|B|^{\frac{d-1}{d}(1+\epsilon)} + |A| + |B|).$$

Cofinal systems of finite sets

Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a system of finite subsets of X which is **cofinal**:

- Any $A \subseteq_{\text{fin}} X$ is a subset of some $B \in \mathcal{C}$.

Theorem (B, Omer Ben-Neria, K, S)

(i) *If* $|X| \geq \aleph_n$*, then* $\mathrm{VC}(\mathcal{C}) > n$*.*

(ii) *This bound is tight for* $n = 1$*:*
There exists such a $\mathcal{C} \subseteq \mathcal{P}(\aleph_1)$ *with* $\mathrm{VC}(\mathcal{C}) = 2$*.*

(iii) *In particular, the existence of a finite VC-dimension cofinal system of finite subsets of* $\mathbb{R}$ *is independent of ZFC.*
(Indeed, $|\mathbb{R}| = \aleph_1$ *and* $|\mathbb{R}| > \aleph_\omega$ *are both consistent.)*

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