$f \in \mathbb{C}[x,y]$

$A, B \subseteq_{fin} \mathbb{C}$ $\quad |A| = n = |B|$

$|f(A,B)| = |\{f(a,b) : a \in A, b \in B\}| \leq n^2$

($=$ for gen^c A, B)

But for $f(x,y) = x+y$, $A = B = [n] := \{0, \dots, n-1\}$

$|f(A,B)| = 2n-1$ ~ linear

sim if f is $\underline{\text{additive}}$: $f(x,y) = g(h(x)+k(y))$ $\quad (g,h,k \in \mathbb{C}[z])$

$(A := h^{-1}([n]),\ B := k^{-1}([n]))$

or mult^{ve} $\quad$ . . . .

$(A := h^{-1}(2^{[n]}) \dots)$

Th^m [Elekes–Rónyai '00, Raz–Sharir–Solymosi '14]:

$\forall A, B \subseteq \mathbb{C}$ s.t. $|A| = |B| = n \gg 0$ $\quad |f(A,B)| \gtrsim n^{1+1/3}$

or f is add^{ve} or mult^{ve}.

// Conj^{re} [Elekes]: $\frac{1}{3} \to \eta$ any $\eta < 1$

Consider $f(x,\bar{y})$, $A \subseteq_{fin} \mathbb{C}$, $B \subseteq_{fin} \mathbb{C}^{|\bar{y}|}$ $\quad |f(A,B)| \geq |A|^{1+\eta}$ unless?

$|F(A)| \leq |F| \cdot |A|$ $\quad$ $\boxed{\text{Naive}}$

Limiting examples

(i) $f(x,\bar{b}) = f(x,\bar{b}')$ $\quad \forall \bar{b}, \bar{b}' \in B$

Generally, $f(A,B) = F(A)$ $\quad F = \{f(x,\bar{b}) : \bar{b} \in B\} \subseteq \mathbb{C}[x]_{\leq d}$

(ii) $|F| < |A|^{\eta} \Rightarrow F(A) < |A|^{1+\eta}$

(iii) $F = \{bx + c : b \in [n],\ c \in [n^{100}]\}$

$A = [n^{100}]$

$F(A) \subseteq [2n^{101}]$

$|F(A)| \leq |A|^{1+1/100}$ $\quad$ F is $1/100$-add^{ve}

// what can we say about F?

Def^n $F \subseteq \mathbb{C}[x]$ is $\underline{\varepsilon\text{-additive}}$ (mult^{ve})

if $\exists g, h \in \mathbb{C}[x]$ $\quad |F \cap \{h(g(x)+c) : c \in \mathbb{C}\}| \geq |F|^{1-\varepsilon}$

$(\cdot)$

Th^m [BZ]: $\forall \varepsilon\ \exists \eta > 0$

$\forall A \subseteq \mathbb{C}$, $F \subseteq \mathbb{C}[x]_{\leq d}$ s.t. $|F| \geq |A|^{\varepsilon} \geq 1/\eta$

either $|F(A)| \geq |A|^{1+\eta}$

or F is ε-add^{ve} or ε-mult^{ve}

Cor^y (Unbalanced ER)

$\forall \varepsilon > 0\ \exists \eta > 0$ $\quad |B| \geq |A|^{\varepsilon} \gg 0 \Rightarrow |f(A,B)| \geq |A|^{1+\eta}$

unless f is add^{ve}/mult^{ve}.

(Rem?)

Pf of Th^n 1

(3)

Fix $\varepsilon > 0$. If no such η,

$(A, F) := \prod_{\mathcal{U}} (A_n, F_n)$ cle to $\eta = 1/n$: $|F_n| \geq |A_n|^{\varepsilon} \geq n$, $\frac{|F(A)|}{|A|^{1+\eta}}$ not ε-η/ve

$A \subseteq K := \mathbb{C}^{\mathcal{U}}$ pseudofinite sets

$F \subseteq K[x]_{<d} = K^d$

$\delta(F) := \delta_A(F) := st(\log_{|A|} |F|) = \lim_{n \to \mathcal{U}} \log_{|A_n|} |F_n| \geq \varepsilon > 0$

(u-lim in $\mathbb{R}$)

$\boxed{\text{Let } A' \subseteq A,\ |A'| = d.\ f(A') \text{ determines } f \in F, \text{ so } \delta(F) \leq d \cdot \delta(F(A)) < \infty}$

$\delta(A) = 1$

Find $a \in A$ s.t. $\delta(a) := \inf_{\phi \in tp(a)} \delta(\phi(K)) = \delta(A)$ working in $(K; +, \cdot, A, F, \ldots)$ "adequate lang"

and $f \in F$ s.t. $\delta(f/a) = \delta(f) = \delta(F)$ ($\deg f > 0$)

$\uparrow$ $a \perp^{\delta} f$

Let $c := f(a)$.

Then $\delta(c) \leq \delta(F(A)) = \delta(A)$

$\Rightarrow c \perp^{\delta} f$ $(\delta(c/f) = \delta(a/f) = \delta(a) = \delta(A) \geq \delta(c))$

$\uparrow$ $a \sim_f c$ $(acl^{ACF}(af) = acl^{ACF}(cf))$

WLOG (working over params)

f is wgp: $\forall d \in K^{<\omega}$ $f \not\perp^{ACF}_{} d \Rightarrow f \not\perp^{\delta} d$

generalising ES

Th^n 2 [BZ]: Sps $a, f, c \in K^{<\omega}$ s.t.

- $0 < \delta(a), \delta(f), \delta(c) < \infty$
- $a \perp^{\delta} f$; $f \perp^{\delta} c$
- $a \sim_f c$; $f \sim_{ACF} cb(ac/f)$; f wgp
- $trd(a) = 1$

Then: $\exists$ comm^{ve} alg^c grp $(G; +)$

$a \sim a' \in G$

$f \sim f' \in G$

$c \sim a' + f' \in G$

$[\rightsquigarrow \varepsilon$-$\eta$/ve$]$

3am

Pf of Th^n 2

(4)

$\tilde{f} := cb(aa'/ff')$ works if f is cgp: $f \not\perp^{ACF} d \Rightarrow \delta(f/d) = 0$

Iterate until get grp conf in ACF

$\rightsquigarrow$ alg^c grp acting on curve

BGT $\Rightarrow$ nilp^{nt}

$\Rightarrow^{*}$ ab^n

$[trd(f) \leq trd(\tilde{f}) \leq \delta(\tilde{f}) \leq \delta(a)]$

$\uparrow$ f cgp; $\uparrow$ a cgp incidence bnd

Else: work over params over which f is cgp

$\rightsquigarrow$ family of ab^n grps

+ family of isogenies

[contradict BGT if doesn't trivialise.

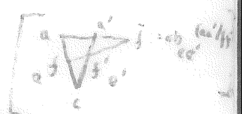
$(\mathbb{G}_m \not\cong Aut^0(G))$

$(\mathbb{G}_a \not\cong \mathbb{G}_m)$]

tr.d$(a) > 1$:

Th^n 2 holds if assume a cgp

(* needs substantial extra arg^{nt}) (Hrushovski? [BSG])



Example: $F \subseteq Aut(\mathbb{C}^2)$

$A \subseteq \mathbb{C}^2$ $|F| \geq |A|^{\varepsilon} \gg 0$

$|F(A)| < |A|^{1+\eta}$

$\Rightarrow$ Ab^n grp

OR A concentrates on curves

$\boxed{\text{v. roughly: } \rightsquigarrow c = e \cdot a + f_1 \quad f = (e, f_1) \quad \rightsquigarrow e \text{ constant}}$ sampled

e.g. $F = \{(x, y) \mapsto (x + \alpha y^2, y + \beta) : \alpha, \beta \in [n]\}$

not ab^n (but $\langle F \rangle$ nilp^{nt} alg^c grp)

$A = [n^3] \times [n]$

$F(A) \subseteq [2n^3] \times [2n]$

WIP: always get nilp^{nt} grp?