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Measurable groups of low dimension

Richard Elwes[*1] and **Mark Ryten**[**2]

[1] 9 Highbury Terrace, Leeds, LS6 4ET
[2] 53A Westbourne Terrace, London W2 3UY, England

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We consider low-dimensional groups and group-actions that are definable in a supersimple theory of finite rank. We show that any rank 1 unimodular group is (finite-by-Abelian)-by-finite, and that any 2-dimensional asymptotic group is soluble-by-finite. We obtain a field-interpretation theorem for certain measurable groups, and give an analysis of minimal normal subgroups and socles in groups definable in a supersimple theory of finite rank where infinity is definable. We prove a primitivity theorem for measurable group actions.

1 Introduction

The study of groups of finite Morley rank has achieved a remarkable depth in recent years, and it is natural to ask to what extent a similar analysis can be done in the context of a supersimple theory of finite rank. However, this has turned out to be problematic in general, and many basic ingredients remain unavailable. For instance, even the possible structure of a group of rank 1 is not yet fully understood. In this paper we prove some elementary results about low-dimensional supersimple groups and group-actions under various extra hypotheses, most notably that of *measurability*: the assumption that the system of definable sets admits a finitary counting measure, as developed in [8, 9, 3, 4].

The results of the latter half of the paper can be seen as paving the way for a classification of asymptotic groups acting on 1-dimensional sets in the mold of [5].

2 Rank-1 unimodular groups

We begin by recalling the definition of an important condition on first order structures.

Definition 2.1 We say that a first-order structure M is *unimodular* if for any parameter-definable sets X and Y in M^{eq}, if $f_1, f_2 : X \longrightarrow Y$ are definable epimorphisms with fibres of constant sizes k_1 and k_2, respectively, then $k_1 = k_2$.

We remind the reader that for supersimple structures of finite rank, and for definable sets (but not necessarily for types, see [13, Example 5.1.15]), S_1-rank, SU-rank, and D-rank coincide. See for instance [7, 6.13 and 6.14]. In this paper we shall use S_1-rank in keeping with [6].

We begin some group-theoretic analysis.

Remark 2.2 Let G be a group. Let $B \subseteq G$ be the set of elements with finite conjugacy classes in G. Then B is a characteristic subgroup of G.

* Corresponding author: e-mail: r.elwes@gmail.com
** e-mail: mjryten@hotmail.com

Proof. To see that B is closed under multiplication, notice that given $b_1, b_2 \in G$ we have

$$(b_1 \cdot b_2)^G \subseteq (b_1)^G \cdot (b_2)^G,$$

where $(b_1)^G$ and $(b_2)^G$ are finite. The rest is clear. $\square$

Recall the following:

Remark 2.3

1. A group G is a *BFC group* (*bounded finite conjugacy-classes*) if all the conjugacy classes of G have finite size, and if there is a finite bound to their sizes.
2. If G is a BFC group, then G' is finite [10, Theorem 3.1].

We can now prove the following generalisation of [9, Theorem 5.15].

Theorem 2.4 *Let G be a group of S_1-rank 1, definable in a unimodular structure. Then G is definably and characteristically (finite-by-Abelian)-by-finite.*

Proof. We assume that our model is sufficiently saturated. If G has infinite exponent, then by saturation there is $g \in G$ of infinite order. But then $C_G(C_G(g))$ is infinite (as $g^n \in C_G(C_G(g))$ for all $n \in \mathbb{N}$) and Abelian, and hence by S_1-rank considerations must be of finite index in G. So assume that G has finite exponent. Hence for any element $y \in G$, the group $\langle y \rangle$ generated by y is definable.

Claim 2.4.1 *G contains a finite non-identity conjugacy class.*

Proof. Suppose not, so all non-identity conjugacy classes are infinite. As G is a disjoint union of its conjugacy classes, S_1-rank 1 forces that there be finitely many, n say.

We define an equivalence relation $\sim$ on the set of conjugacy classes:

$$C_1 \sim C_2 :\Leftrightarrow (\exists y_1 \in C_1, y_2 \in C_2)(\langle y_1 \rangle = \langle y_2 \rangle).$$

Suppose that there are n_2 different $\sim$-classes of conjugacy classes. For a conjugacy class y^G we denote its $\sim$-equivalence class by $\widetilde{y^G}$.

Now $\langle y_1 \rangle$ has $\Phi(|\langle y_1 \rangle|)$ many distinct generators, where Φ is Euler's totient function. But $N_G(\langle y_1 \rangle)$ acts transitively on the set of generators of $\langle y_1 \rangle$ which are conjugate to y_1, and so by the Orbit-Stabilizer Theorem there are exactly $|N_G(\langle y_1 \rangle)| \,/\, |C_G(\langle y_1 \rangle)|$ of these. Notice too that if $\langle y_1 \rangle = \langle y_2 \rangle$, then

$$|N_G(\langle y_1 \rangle)| \,/\, |C_G(\langle y_1 \rangle)| = |N_G(\langle y_2 \rangle)| \,/\, |C_G(\langle y_2 \rangle)|.$$

So $\Phi(|\langle y_1 \rangle|) = |\widetilde{y_1^G}| \cdot \dfrac{|N_G(\langle y_1 \rangle)|}{|C_G(\langle y_1 \rangle)|}$.

Picking representatives x_i, notice that for each conjugacy class x_i^G, we have that there is a map $f : G \longrightarrow x_i^G$ given by $f(g) := x_i^g$ which has constant fibre of size $K_i := |C_G(x_i)|$. Put $K := \operatorname{lcm}\{K_i \mid i \le n\}$. Now let

$$\{G_{ir} \mid 1 \le i \le n, 1 \le r \le \frac{K}{K_i}\}$$

be a family of pairwise disjoint copies of G. We obtain a new map $f_1 : \bigsqcup_{i=1}^{n} \bigsqcup_{r=1}^{\frac{K}{K_i}} G_{ir} \longrightarrow G$ by $f_1(g) := x_i^g$ for $g \in G_{ir}$, which has constant fibre of size K.

Of course we may write $\{G_{ir} \mid 1 \le i \le n, 1 \le r \le \frac{K}{K_i}\}$ as

$$\{G_{ijr} \mid 1 \le i \le n_2, 1 \le j \le |\widetilde{y_i^G}|, \text{and } 1 \le r \le \frac{K}{K_i}\}$$

and f_1 as $f_1 : \bigsqcup_{i=1}^{n_2} \bigsqcup_{j=1}^{|\widetilde{y_i^G}|} \bigsqcup_{r=1}^{\frac{K}{K_i}} G_{ijr} \longrightarrow G$. But there is another map $f_2 : \bigsqcup_{i=1}^{n_2} \bigsqcup_{j=1}^{|\widetilde{y_i^G}|} \bigsqcup_{r=1}^{\frac{K}{K_i}} G_{ijr} \longrightarrow G$, namely $f_2(g) = g$. Now we use the assumption of unimodularity to get

$$K = \sum_{i=1}^{n_2} \frac{\Phi(|\langle y_i \rangle|) \cdot |C_G(\langle y_i \rangle)|}{|N_G(\langle y_i \rangle)|} \cdot \frac{K}{K_i} \quad \text{and} \quad 1 = \sum_{i=1}^{n_2} \frac{\Phi(|\langle y_i \rangle|) \cdot |C_G(\langle y_i \rangle)|}{K_i \cdot |N_G(\langle y_i \rangle)|}.$$

If the exponent of G is odd, then all numbers in the above expression are odd except $\Phi(|\langle y_i \rangle|)$ which yields a contradiction. So the exponent of G must be even, and hence G contains an involution, g, say, with conjugacy class C, say. By assumption C is infinite. Define $S(g) := \{x \in G \mid x^g = x^{-1}\}$. Then $g^G \cdot g \subseteq S(g)$ (as g is an involution). Now if k is the maximum size of a centralizer of a non-identity element, then $\{S(h) \mid h \in C\}$ is a $(k+2)$-inconsistent family, since if there were $x \in S(h_1) \cap \cdots \cap S(h_{k+2})$, then there would be $k+1$ distinct elements $h_1 h_i$, for all of which $x^{h_1 h_i} = ((x^{-1})^{h_i}) = x$, so every $h_1 h_i$ centralizes x, which is impossible by the maximality of k. But then $\{S(h) \mid h \in C\}$ contradicts G having S_1-rank 1. □ (Claim 2.4.1)

Let N be the union of all the finite conjugacy classes in G. By Remark 2.2, N is a characteristic subgroup. If there are only finitely many finite conjugacy classes, then N is finite and hence definable, and G/N has only infinite conjugacy classes. Moreover G/N certainly has S_1-rank 1, and is also unimodular (as unimodularity is a condition on G^{eq}). But then G/N contradicts the claim.

Hence there are infinitely many finite conjugacy classes in G. By compactness and saturation, they have bounded size as otherwise G would contain infinitely many infinite conjugacy classes contradicting S_1-rank 1. Thus N is definable and infinite. By S_1-rank 1 then, $[G : N]$ is finite, and moreover N is a BFC group. Therefore by Remark 2.3, N' is finite, and G is (finite-by-Abelian)-by-finite. □

3 Indecomposability theorem and field interpretation theorem

Before proceeding with our analysis of 2-dimensional groups, we will observe that two important theorems from the study of groups of finite Morley rank have analogues in our context. First though we review some terminology.

The notion of a *measurable* structure is a strengthening of unimodularity, in which each definable set is assigned a dimension and measure. It implies supersimplicity of finite rank [8, Corollary 3.6]. An *asymptotic class* is a class of finite structures, in which the cardinality of the definable sets is governed by their dimension and measure. Any infinite ultraproduct from an asymptotic class is measurable. We refer the reader to [8, 9, 3, 4] for further background on these notions.

In the context of an asymptotic class $\mathcal{C}$ of groups we may refer to an *asymptotic group* G. This term shall mean that G is a non-principal ultraproduct of members of $\mathcal{C}$.

We write $M = \prod_{i \in I} M_i/\mathcal{U}$ to mean that M is the infinite ultraproduct of $\{M_i \mid i \in I\}$ given by the non-principal ultrafilter $\mathcal{U}$ on I. We may refer to a structure M_i as a *component* of the ultraproduct.

We recall this definition from [6, Definition 4.2].

Definition 3.1 A first order theory is an S_1*-theory* if every formula has finite S_1-rank, and if for every definable set P where $S_1(P) = m$ and every formula $\varphi(x, b)$, the set $\{b \mid S_1(\varphi(M, b) \cap P) = m\}$ is definable, uniformly in the parameters for P.

It is immediate that the theory of a measurable structure in which dimension and S_1-rank coincide is an S_1-theory, which is a fact we shall use in the next section, particularly in the light of the following lemma.

Lemma 3.2 *Let M be a measurable structure, where* $\dim(M) = S_1(M) \leq 2$. *Then for any definable set X, we have* $\dim(X) = S_1(X)$.

P r o o f. (It will be convenient to work with the formulation of D-rank, see for instance [13, Definition 5.1.13]. However, bearing in mind the remark following Definition 2.1, we shall abuse notation and refer to it as S_1-rank.) Now by [8, Corollary 3.6] we know that $S_1(X) \leq \dim(X)$. We show that $S_1(X) \geq \dim(X)$. Suppose $X \subseteq M^n$. We proceed by induction on n. For $n = 1$, the case where $\dim(M) = S_1(M) = 2$ and $\dim(X) = 2$ is the only one in which there is anything to prove. Suppose for a contradiction that $S_1(X) = 1$. Then X divides over $\emptyset$, and so there exists a k-inconsistent infinite family of conjugates of X, say $\{X_i \mid i \in I\}$, but by [8, Lemma 3.5] this contradicts $\dim(M) = 2$. Now suppose $X \subseteq M^n$. Consider the projection $\pi(X)$ onto the first coordinate. By induction we know that for each $a \in \pi(X)$ we have $\dim(X_a) = S_1(X_a)$, where X_a is the fibre over a. For each d' we put

$$Y_{d'} := \{a \in \pi(X) \mid \dim(X_a) = d'\}.$$

Now we may choose d such that $d + \dim(Y_d)$ is maximal. Say $\dim(Y_d) = e$. Of course we have $e \in \{0, 1, 2\}$, and $S_1(Y_d) = e$. Also $\dim(X) = d + e$, by the definition of measurability. Put $Z := \bigcup_{a \in Y_d} X_a$.

If $e = 0$, then clearly $S_1(Z) = d$. If $e = 1$, then the $\{X_a \mid a \in Y_d\}$ witness that $S_1(Z) \geq d + 1$. If $e = 2$, then this must be witnessed by some k-inconsistent family $\{A_i \mid i \in \mathbb{N}\}$ of subsets of Y_d of S_1-rank 1. Put

$$B_i := \bigcup_{a \in A_i} X_a.$$

Then for each i the fibres X_a witness that $S_1(B_i) \geq d + 1$, and moreover the family $\{B_i \mid i \in \mathbb{N}\}$ then witnesses that $S_1(Z) \geq d + 2$.

In all cases this shows that $S_1(X) \geq d + e$. □

One fact that we will make repeated use of without comment is this.

Lemma 3.3 *Let G be a measurable group, and $H_1 \leq H_2$ definable subgroups, with*

$$(\dim, \mathrm{meas})(H_i) = (d_i, \mu_i) \quad \textit{and} \quad (\dim, \mathrm{meas})(\mathrm{Cos}(H_1 : H_2)) = (e, \nu)$$

(*where* $\mathrm{Cos}(H_1 : H_2)$ *is the coset-space of* H_2 *in* H_1). *Then* $(d_2, \mu_2) = (d_1 + e, \mu_1 \cdot \nu)$.

P r o o f. This is immediate, since $(\dim, \mathrm{meas})(\mathrm{Cos}(H_1 : H_2))$ is well-defined by [9, Proposition 5.10]. □

The following is another condition on a first order theory which we will need.

Definition 3.4 Let T be a first order theory. We say that *infinity is definable in* T, or that T *eliminates* $\exists^\infty$, if for every $M \vDash T$ and every formula $\varphi(\bar{x}, \bar{y})$, the set $\{\bar{a} \in M^m \mid \varphi(M^n, \bar{a})$ is finite$\}$ is $\emptyset$-definable.

It is immediate that in both measurable and S_1-theories infinity is definable.

We now discuss an analogue of Zilber's Indecomposability Theorem (see for instance [1, Theorem 5.26]) that was proved by Hrushovski in the context of S_1-theories (see [6, Theorem 7.1]). A more general version has been obtained by Wagner for supersimple theories (see [13, Theorem 5.5.4]). The following is the version we will need.

Remark 3.5 Let G be a group definable in a supersimple structure of finite rank. Let $\{X_i \mid i \in I\}$ be a collection of definable subsets of G. Then there exists a definable subgroup H of G such that:

(i) $H \leq \langle X_i \mid i \in I\rangle$ and every element of H is a product of a bounded finite number of elements of the X_i's and their inverses.

(ii) X_i/H is finite for each i.

If each X_i is $\mathrm{acl}(\emptyset)$-definable, then so is H.

If the collection of sets X_i is invariant under conjugation, then H may be chosen normal in G. More generally, if the collection of X_i is Σ-invariant for some collection of definable automorphisms Σ, then H may be chosen to be Σ-invariant, too.

P r o o f. This is easily derived from Wagner's more general result [13, Theorem 5.4.5], which yields the existence of a type-definable group of the desired kind. [13, Theorem 5.5.4] and compactness then give a definable group, exactly as in the proof of the corresponding result for S_1-theories [6, Theorem 7.1]. The final clause is a consequence of [13, Remark 5.4.7]. □

Now we obtain a version of Zilber's Field Interpretatation Theorem (see [1, Theorem 9.1]) applicable in our context. This will not be used during the remainder of this paper.

Proposition 3.6 *Let G be a group definable in a supersimple structure of finite rank in which infinity is definable. Suppose that $G = A \rtimes H$, where A and H are each Abelian, definable in G, and 1-dimensional. Suppose A has no proper non-trivial G-definable subgroups which are normal in G, and $C_H(A) = \{1\}$. Then the following wonderful things happen:*

1. *The subring $K = \mathbb{Z}[H]/\mathrm{ann}_{\mathbb{Z}[H]}(A)$ of $\mathrm{End}(A)$ is a definable measurable field; in fact, there is an integer l such that every element of K can be represented as the endomorphism $\sum_{i=1}^{l} h_i$ $(h_i \in H)$.*

2. *$A \cong K^+$. Also H is isomorphic to a subgroup J of K^* and the conjugation action of H on A is its multiplication action on K.*

3. *If G is an ultraproduct of finite groups, then K is a pseudofinite field.*

P r o o f. Let B be the union of the finite orbits under the action of H on A. Then B is a G-normal subgroup of A, and as infinity is definable, B is definable. Hence either $B = \{1\}$ or $B = A$. If $B = A$, it follows that for each $a \in A$, its point-stabilizer in H, that is $C_H(a)$, has finite index in H. Thus for any n and any $a_1, \dots, a_n \in A$, the group $C_H(a_1, \dots, a_n)$ is non-trivial. Thus by compactness, $C_H(A)$ is non-trivial, a contradiction. Thus we must have $B = \{1\}$, and thus the action of H on $A \setminus \{0\}$ has only 1-dimensional orbits, and necessarily there are only finitely many of them.

1. We consider the group ring $\mathbb{Z}[H]$ as a ring of endomorphisms of A, extending the conjugation action of H by linearity. We will use the notation $\cdot$ for the action of $\mathbb{Z}[H]$ or its quotients/subrings on A. Let $r \in \mathbb{Z}[H]$. Say

$$r = \sum h_i \quad (h_i \in H).$$

Since H is Abelian, $\ker(r)$ and $\mathrm{im}(r)$ of the endomorphism $r : A \longrightarrow A$ are both definable H-normal subgroups of A. Thus either $\ker(r) = A$, in which case $r \in \mathrm{ann}_{\mathbb{Z}[H]}(A)$, or $\ker(r) = \{0\}$, and $\mathrm{im}(r) = A$ and r is an automorphism of A.

Let $R = \mathbb{Z}[H]/\mathrm{ann}_{\mathbb{Z}[H]}(A)$. We have shown that R is a ring of automorphisms of A.

Now let $a \in A \setminus \{0\}$. Let W be the orbit of a under the conjugation action of H. From above we know that W is a definable set of dimension 1. Applying Remark 3.5 to W we may find a definable 1-dimensional subgroup C of A, where $C \leq \langle W \rangle$. Thus C is a definable subgroup of A of finite index, and further we may demand that C is normalised by H. Thus $C = \langle W \rangle = A$. By compactness there is an integer l and $A = W + \dots + W$ (l times). We let

$$K = \{\textstyle\sum_{i=1}^{l} h_i \mid h_i \in H\}/\mathrm{ann}_{\mathbb{Z}[H]}(A).$$

So $K \subseteq R \subseteq \mathrm{End}(A)$.

Suppose $\lambda \in R \setminus \{0\}$ and $b \in A$ such that $b = \lambda \cdot a$. So there is $\zeta \in K$, and $\zeta \cdot b = a$. So

$$\zeta\lambda - 1 \in R \quad \text{and} \quad (\zeta\lambda - 1) \cdot a = 0.$$

Since R is a ring of automorphisms we deduce $\zeta\lambda = 1$. So in fact R is a field. Furthermore, for every $\lambda \in R$ there is $\zeta \in K$ and $\zeta = \lambda^{-1}$. Since R is closed under inverses, we deduce $K = R$. This concludes the proof of 1.

2. Now define $i_a : K \longrightarrow A$ by $i_a(\lambda) = \lambda \cdot a$. By the construction of K we know that i_a is an additive isomorphism from K onto A. To make this an isomorphism of fields we must define the correct multiplication on A. We use the symbol $\otimes$ to define the multiplication on A. Then the definition is: if $b = \lambda \cdot a$ and $c = \zeta \cdot a$, then

$$b \otimes c := i_a(\lambda\zeta).$$

It is easily verified that i_a is now an isomorphism of fields sending H to a finite index subgroup of $A^{\otimes}$.

3. Say $G = \prod_{i \in I} G_i/\mathcal{U}$. By Łos' Theorem, there is $U \in \mathcal{U}$ where the formulae which define the field structure on K also do so on every G_i for which $i \in U$. So K is in fact a non-principal ultraproduct of finite fields, and thus a pseudofinite field. □

4 2-dimensional asymptotic groups

We now proceed to discuss groups of S_1-rank 2. In all that follows, when we speak of a class $\mathcal{C}$ of groups being *P-by-bounded* we shall mean that for each $G \in \mathcal{C}$ there is $N \lhd G$, where N has property P, and $|G : N|$ is finite and bounded as G ranges across $\mathcal{C}$.

We shall prove (Theorem 4.7 below) that a 2-dimensional asymptotic class of groups is soluble-by-bounded. We will need the following, which is unknown at the level of measurable groups.

Remark 4.1 (Ryten) There is no 2-dimensional weak asymptotic class of simple groups.

Ryten shows that any class of finite simple groups of fixed Lie type and Lie rank is uniformly bi-interpretable with a class of finite difference fields known to form an asymptotic class. It follows that for $r \in \mathbb{N}$, the class of simple groups of Lie rank $\leq r$ is an aymptotic class. Then using the classification of the finite simple groups he shows that every weak asymptotic class of simple groups lies within a family of this form. As no infinite subclass of any of these classes can have dimension equal to 2, the result follows. For more details see [11] or [12].

We will say that a group G is *n-soluble* if it is soluble of derived length n. Similarly G is *n-nilpotent* if it is nilpotent of class n. Now we need a group-theoretic fact.

Lemma 4.2

1. *A finite-by-(n-soluble) group is definably (($n+1$)-soluble)-by-finite.*
2. *A finite-by-Abelian group is (2-nilpotent)-by-finite.*

Proof. Suppose G is finite-by-(n-soluble). Say that $F \triangleleft G$ is finite and G/F is n-soluble. So G acts on F by conjugation. Let $H := C_G(F)$. Then H is definable and $|G : H|$ is finite. Now $H^{(n)} \leq G^{(n)} \cap H$. Since

$$\{1\} = (G/F)^{(n)} = G^{(n)}/F,$$

we have $G^{(n)} \leq F$ and in particular $H^{(n)} \leq F$. But H commutes with F, so $H^{(n)}$ is Abelian and therefore H is $(n+1)$-soluble. The result follows since H is of finite index in G.

In the case where G/F is Abelian, i.e. $n = 1$, $H' \leq F$ again, but as $H := C_G(F)$, we obtain $[H, H'] = \{1\}$ as required. □

Remark 4.3 Recall that for any type-definable group G and any set A of parameters, *the A-connected component of G*, denoted G^0_A, is the intersection of all type-definable subgroups of G over A of bounded index. [13, Lemma 4.1.11] tells us that in the context of a simple theory, G^0_A is a type-definable normal subgroup of index at most $2^{|T(A)|}$ in G.

Lemma 4.4 *Let G be a group with S_1-theory and S_1-rank 2, and with no definable S_1-rank 1 normal subgroups. Then at least one of the following holds: G is definably (2-nilpotent)-by-finite; or G contains a minimal 2-dimensional definable subgroup, which is normal in G.*

Proof. Let B be the set of finite conjugacy classes of G. By Remark 2.2, B is a normal subgroup of G, and by the fact that infinity is definable in S_1-theories, B is definable. If $S_1(B) = 2$, then G is (finite-by-Abelian)-by-finite, again by Remark 2.3. By Lemma 4.2 such a group is, in fact, (2-nilpotent)-by-finite. Thus we may assume that $S_1(B) \neq 2$. Since $S_1(B) \neq 1$ we conclude that B is finite.

Suppose for a contradiction that there is an infinite descending chain $G = W_0 > W_1 > W_2 > \cdots$, where for each $i > 0$ we have $W_i < G$, and $\dim(W_i) = 2$, and W_i is definable over $\bar{a}_i$ say. Notice that W_i is of finite index in G. Let $A := \bigcup_{i\in\mathbb{N}} \bar{a}_i$.

In the notation of Remark 4.3, G^0_A is of infinite index in G, as for all $i \in \mathbb{N}$, $G^0_A \leq W_i$. Also by Remark 4.3, G^0_A is of bounded index in G. We have assumed a fair degree of saturation on G and we conclude that G^0_A is infinite. In particular, we may pick $\gamma \in G^0_A \setminus B$, and consider $\Gamma := \langle \gamma^G \rangle$. Notice that Γ is infinite since $\gamma \notin B$. Also $\Gamma \triangleleft G$, and $\Gamma \leq G^0_A$.

Now by Remark 3.5 there is a definable $N \leq \Gamma$, where $N \triangleleft G$ and γ^G/N is finite. Hence N is infinite, so we have $S_1(N) \geq 1$. But $N \leq G^0_A$, so $[G : N]$ is infinite, and thus $S_1(N) = 1$, which is impossible.

Hence every such sequence $G = W_0 > W_1 > W_2 > \cdots$ must stabilize at some finite stage. Thus there is a minimal 2-dimensional definable subgroup of G. Since there must be a normal subgroup of G of finite index inside this minimal subgroup, it follows that it itself is normal in G. □

Lemma 4.5 *Let G be a unimodular group with S_1-theory and S_1-rank 2. If G contains a definable S_1-rank 1 normal subgroup, then G is definably (4-soluble)-by-finite.*

Proof. Suppose N is a definable normal subgroup and $\dim(N) = 1$. Then by Theorem 2.4 there is a definable and characteristic subgroup H of N which is finite-by-Abelian and of finite index in N.

Now we consider the quotient G/N. Suppose $\pi : G \longrightarrow G/N$ is the quotient homomorphism.

Now the quotient G/N is a unimodular S_1-rank 1 group, and by Theorem 2.4 it is definably a (finite-by-Abelian)-by-finite group. Again, all pieces may be taken to be characteristic. We denote the bottom finite-by-Abelian piece by J^*. Now let $J := \pi^{-1}(J^*)$, so that $J^* = J/N$. Notice that $(J^*)'$ is finite, and that $\pi^{-1}((J^*)') = J'N$. Then we have the following tower of *normal* subgroups of G:

$$H' \leq H \leq N \leq J'N \leq J \leq G.$$

Now consider J/H, which is a finite-by-Abelian group. J/H is an infinite group acting by conjugation on the finite set $J'N/H$. So there is a definable subgroup of finite index $J_1/H \leq J/H$ which is the kernel of this action. Thus J_1/H commutes with all the elements of $J'N/H$. But now consider J_1'/H: since

$$J_1'/H \leq J'N/H \cap J_1/H$$

it must be Abelian. This shows $J_1''/H = 1$ so $J_1'' \leq H$. So $J_1''' \leq H'$ and H' is finite. By considering the conjugation action of J_1 on J_1''' we see that there is a definable subgroup J_2 of J_1, with $[J_1 : J_2] < \infty$ and such that J_2 commutes with J_1'''. So $J_2''' \subseteq J_1''' \cap J_2$ and thus J_2''' must be Abelian. So J_2 is 4-soluble and definable, and it is a finite index subgroup of G. □

Lemma 4.6 *Let G be a unimodular group with S_1-theory and S_1-rank 2. Then either G is definably (4-soluble)-by-finite, or G is definably (finite-by-simple)-by-finite, and therefore interprets a simple group of S_1-rank 2.*

P r o o f. Suppose G has S_1-theory and is of S_1-rank 2. If G has a 1-dimensional definable normal subgroup, then by Lemma 4.5, G is (4-soluble)-by-finite. So suppose that G has no such subgroup. Then by Lemma 4.4, G is either 2-nilpotent-by-finite (which certainly implies (4-soluble)-by-finite), or G contains a minimal 2-dimensional definable subgroup H, where $H \lhd G$.

Now if H contains a 1-dimensional definable normal subgroup, then by Lemma 4.5, H is (4-soluble)-by-finite, and so G is ((4-soluble)-by-finite)-by-finite, which implies (4-soluble)-by-finite. Otherwise H contains no 1-dimensional definable normal subgroups.

We consider N: the subgroup of H consisting of the finite conjugacy classes of H. Again N is definable and normal and contains each finite normal subgroup of H. So either $\dim(N) = 2$ in which case just as in Lemma 4.4, H is 2-nilpotent-by-finite (which would again yield the result), or $\dim(N) = 0$. In this case then H/N is measurable (by [9, Proposition 5.10]), 2-dimensional, and contains no proper definable normal subgroups of dimension 1 or 2 (since H does not), and no definable normal subgroups of dimension 0 (since all such subgroups of H lie inside N).

Therefore H/N is definably simple. Obviously H/N is not Abelian as it contains infinite conjugacy classes. So we conclude by [6, Corollary 7.4] that H/N is simple. □

Theorem 4.7 *Let $\mathcal{C}$ be a 2-dimensional asymptotic class of groups. There exists a positive integer $m \in \mathbb{N}$ and a finite set of $\mathcal{L}$-formulae $\{\varphi_1(x), \ldots, \varphi_n(x)\}$ in one variable such that for every $G \in \mathcal{C}$, one of the $\varphi_i(x)$ defines a normal subgroup of index less than m in G which is 4-soluble.*

P r o o f. Suppose that for each finite set of $\mathcal{L}$-formulae in one variable, no such m exists. Enumerate the $\mathcal{L}$-formulae in one variable: $\{\varphi_i(x) \mid i \in \mathbb{N}\}$. Then for every $n, m \in \mathbb{N}$ there is $G_{n,m} \in \mathcal{C}$ so that none of $\varphi_0, \ldots, \varphi_n$ defines a 4-soluble subgroup of index at most m in $G_{n,m}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let

$$G := \prod_{m \in \mathbb{N}} G_{m,m}/\mathcal{U}.$$

Now if G has S_1-rank 1, then we may apply Theorem 2.4 to obtain a contradiction. Otherwise G has S_1-rank 2, and therefore G has S_1-theory by Lemma 3.2. Moreover G is unimodular. By [6, Section 7], being 4-soluble is a definable property. Thus G is not definably (4-soluble)-by-finite: if $\varphi_j(x)$ defined a 4-soluble subgroup of index r in G, then by Łos' Theorem this would have to hold for unboundedly many $G_{m,m}$, and in particular it would hold for some $G_{m,m}$, where $m \geq \max\{j, r\}$ which is impossible. Thus, by Lemma 4.6, G interpets a 2-dimensional simple group H.

Then by Łos' Theorem again there is $U \in \mathcal{U}$ so that for all $i \in U$, the interpretation of H in G carries down to interpret a 2-dimensional group H_i in G_i. Now if for some $U' \in \mathcal{U}$ and all $i \in U'$ there was a proper non-trivial normal subgroup $N_i \lhd H_i$, then (setting $N_i := \{1\}$ for $i \notin U'$) we would obtain a proper non-trivial normal subgroup $N = \prod_{i \in \mathbb{N}} N_i/\mathcal{U} \lhd H$. Hence there is an infinite family $\mathcal{H}$ of 2-dimensional simple groups, and these comprise a weak asymptotic class. But by Remark 4.1, no such class exists. □

At the level of measurable groups we get a dichotomy: either every 2-dimensional measurable group is 4-soluble-by-finite, or there exists a simple one. The obstacle is a measurable version of Remark 4.1 which we have not yet got, though some structural results for 2-dimensional measurable simple groups exist.

5 Minimal normal subgroups and socles in groups of finite rank

For this section our standing assumption will be that G is a definable group in a saturated supersimple structure of finite rank. When we come to Proposition 5.5, however we will also need to assume that infinity is definable. The following will be our main concern fo this section:

Definition 5.1 A *minimal* (*definable*) *normal subgroup of* G is a non-trivial (definable) normal subgroup of G properly containing no other non-trivial (definable) normal subgroup of G. *The socle of* G is the group generated by the minimal normal subgroups of G.

Lemma 5.2 *Suppose that* G *has no finite conjugacy classes. Then minimal normal subgroups of* G *exist and are definable.*

P r o o f. Suppose that H is a normal subgroup of G. Let $x \in H$. Then we may apply Remark 3.5 to the infinite conjugacy class x^G. Thus there is a non-trivial definable normal subgroup $N \triangleleft G$ with $N \leq H$. In the case that H is minimal normal, we must have that $N = H$, and therefore H is definable.

Now pick W a definable normal subgroup of G of minimal dimension, and suppose that

$$W = W_0 > W_1 > W_2 > \cdots$$

is a descending chain of definable normal subgroups of G. Say W_i is defined over $\bar{a}_i$. Let $A := \bigcup_{i \in \mathbb{N}} \bar{a}_i$.

Then look at W^0_A: the A-connected component of W. This is type definable and of infinite index in W, since for all $i \in \mathbb{N}$, $W^0_A \leq W_i$. Also by Remark 4.3, W^0_A is of bounded index in W. Assuming a fair degree of saturation, we may conclude that W^0_A is infinite. In particular, we may pick $\gamma \in W^0_A$, where γ^G is infinite. Now consider $\Gamma := \langle \gamma^G \rangle$. Then Γ is infinite, and $\Gamma \triangleleft G$, and $\Gamma \leq W^0_A$ since $\gamma \in W^0_A$ and $W^0_A \triangleleft G$.

Now by Remark 3.5 there is a definable $N \leq \Gamma$, where $N \triangleleft G$ and $\{x^{-1}\gamma x \mid x \in G\}/N$ is finite, and hence N is infinite. But $N \leq W^0_A$, so $[W : N]$ is infinite, and thus $\dim(N) < \dim(W)$, which is impossible.

Thus any such sequence $W_0 > W_1 > W_2 > \cdots$ must eventually stabilize, at which stage we obtain a minimal normal subgroup. □

Lemma 5.3

1. *Any two distinct minimal normal subgroups centralise each other.*

2. *If in addition* G *has no non-trivial finite conjugacy classes, then the socle of* G *is definable, and is a finite direct product of minimal normal subgroups.*

P r o o f.

1. Suppose H and K are two distinct minimal normal subgroups. So $[H, K] \triangleleft G$ and $[H, K] \subseteq H \cap K$, so by minimality $[H, K] = 1$.

2. We first show

Claim 5.3.1 *Suppose* $\{H_i \mid 1 \leq i \leq n\}$ *is a set of minimal normal subgroups of* G. *There is* $J \subseteq \{1, 2, \ldots, n\}$ *such that* $\langle H_i \mid 1 \leq i \leq n \rangle = \prod_{j \in J} H_j$ (*here* $\prod$ *means direct product*).

P r o o f. We prove this inductively on n. Suppose $m < n$ and there is $J_m \subseteq \{1, 2, \ldots, m\}$ such that

$$\langle H_i \mid 1 \leq i \leq m \rangle = \prod_{j \in J_m} H_j.$$

Consider $H_{m+1} \cap \prod_{j \in J_m} H_j$. Since it is a normal subgroup of G contained in H_{m+1}, this intersection is either trivial or is H_{m+1}. □ (Claim 5.3.1)

Let S be the socle of G. Since G has no finite conjugacy classes, each minimal normal subgroup is definable and must have dimension at least 1. Now there must be a collection $\{H_i \mid 1 \leq i \leq n\}$ of definable, minimal normal subgroups of G such that $S = \prod_{i=1}^{n} H_i$, since if not, then by Claim 5.3.1 we can find direct products of this form for arbitrarily large n. But such a subgroup must have dimension at least n, which is impossible. □

The following is a standard result.

Remark 5.4 Suppose that A is an Abelian group with no non-trivial proper definable characteristic subgroups. Then either A is an elementary p-group for some prime p, or A is torsion-free and divisible.

We are now in a position to classify the minimal definable normal subgroups:

Proposition 5.5 *Suppose now that infinity is definable in G. Let M be a minimal definable normal subgroup of G, and suppose that M is infinite. Then either* 1.*,* 2.*, or* 3. *holds:*

1. *M is an elementary p-group.*
2. *M is a $\mathbb{Q}$-vector-space.*
3. *M is a minimal normal subgroup of G, and is a finite direct product of isomorphic, definable, simple groups.*

Proof. The subgroup of finite conjugacy classes of M is normal in G and (since infinity is definable) is definable. So it is either trivial or the whole of M. Suppose it is the latter. Then by Remark 2.3, we find that M is finite-by-Abelian. But M' is characteristic and by Remark 2.3 it is definable, so $M' = \{1\}$. So M is Abelian. It has no definable characteristic subgroups, so by Remark 5.4 it is either of type 1. or type 2.

Now suppose the subgroup of finite conjugacy classes of M to be trivial. By Lemma 5.2, minimal normal subgroups of G exist and are definable, thus M is one such. Lemma 5.2 applied to M also shows that the minimal normal subgroups of M are definable infinite groups. Let $T \triangleleft M$ be one such. Let $x \in G$ and let $T_1 = xTx^{-1}$. Since conjugation by x is an automorphism of M, it follows that T_1 is also a minimal normal subgroup of M. Therefore $T_1 = T$ or $T_1 \cap T = \{1\}$, so $[T, T_1] = \{1\}$. Just as in Lemma 5.3 this shows that for any finite collection $\{T_j \mid j \in I\}$ of distinct G-conjugates of T, there is a subset $J \subseteq I$ such that

$$\langle \{T_j \mid j \in I\} \rangle = \textstyle\prod_{j \in J} T_j.$$

Since $\dim(T) \geq 1$ the possible number of G-conjugates in such a direct product has a finite upper bound. Thus

$$\langle aTa^{-1} \mid a \in G \rangle = \textstyle\prod_{i=1}^{n} a_i T a_i^{-1}$$

for some n and $a_1, \ldots, a_n \in G$. So let

$$T_0 = \langle aTa^{-1} \mid a \in G \rangle.$$

Thus, clearly T_0 is definable, normal in G, and contained in M. We conclude $T_0 = M$.

It only remains to show that T is simple. Say $S \triangleleft T$. But there is a definable group Y such that $M = T \times Y$ and so $S \triangleleft M$. Since T is a minimal normal subgroup of M it follows that $S = T$ or $S = 1$. □

6 Measurable group actions

Suppose (G, X) is a structure consisting of a group G, a set X, and a group action map $F : G \times X \longrightarrow X$. Formally we consider a language $\mathcal{L} = \{G, X, F, m, 1, i\}$, where G and X are unary predicates, i is a unary function, m is a function with arity 2, and F is a function with arity 2. We provide axioms stating that the elements with predicate G form a group with respect to the symbols $m, i, 1$. The function F, when restricted to $G \times X$, is axiomatised to yield a group action of G on X. We have no concern with what the operations do elsewhere. For instance, we might consider only models where on the ranges where the various functions should not be defined they send all elements to 1.

We call such a first order structure a *group action*. Instead of using the function symbol F, we write $F(g, x)$ as $g \cdot x$, for $g \in G$ and $x \in X$.

A *measurable group action* will mean a group action which is measurable as a first order structure. Similarly, an *asymptotic class of group actions* is an asymptotic class whose members are finite group actions, and an *asymptotic group action* will be an infinite ultraproduct of members of such a class.

If G is a group acting on a set X, and $x \in X$, then the stabilizer of x in G will be denoted by G_x. Similarly if $Y \subseteq X$, then the setwise-stabilizer of Y in G is denoted by G_Y.

Primitive group actions play a central role in permutation group theory. Many questions about general actions can be reduced to the primitive case. Recall that an action (G, X) is *primitive* if there is no non-trivial proper G-congruence on X, or equivalently if every point-stabilizer G_x is a maximal subgroup.

We will say that a group action is *definably primitive* if for each $x \in X$ the group G_x is definably maximal, i. e. there is no definable group H where $G_x < H < G$.

Proposition 6.1 *Let G be a measurable group acting definably and transitively on a definable set X. Define $\sim$ on X by*

$$x \sim y :\Leftrightarrow [G_x : G_x \cap G_y] < \infty.$$

Then $\sim$ is a definable G-congruence.

Proof. First observe that as G is measurable, there are only finitely many finite values which $[G_x : G_x \cap G_y]$ can take, so $\sim$ is definable.

$\sim$ is obviously reflexive.

$\sim$ is symmetric: for any $x \in X$ and $a \in G$, $\mu(G_x) = \mu(aG_xa^{-1}) = \mu(G_{ax})$, so by transitivity of the action, $\mu(G_x) = \mu(G_y)$ for all $x, y \in X$. Also if $x \sim y$,

$$\mu(G_x \cap G_y) \cdot [G_x : G_x \cap G_y] = \mu(G_x) = \mu(G_y) = \mu(G_x \cap G_y) \cdot [G_y : G_x \cap G_y],$$

so $[G_x : G_x \cap G_y] = [G_y : G_x \cap G_y] < \infty$.

$\sim$ is transitive: suppose that $x \sim y$, and $y \sim z$. Then

$$[G_y \cap G_x : G_y \cap G_z \cap G_x] \leq [G_y : G_y \cap G_z] < \infty.$$

So

$$[G_x : G_x \cap G_y \cap G_z] = [G_x : G_x \cap G_y] \cdot [G_x \cap G_y : G_x \cap G_y \cap G_z] < \infty.$$

But also

$$[G_x : G_x \cap G_y \cap G_z] = [G_x : G_x \cap G_z] \cdot [G_x \cap G_z : G_x \cap G_y \cap G_z],$$

so $[G_x : G_x \cap G_z] < \infty$.

$\sim$ is a G-congruence: suppose $x \sim y$, so $[G_x : G_x \cap G_y] < \infty$. So for any $g \in G$,

$$[gG_xg^{-1} : gG_xg^{-1} \cap gG_yg^{-1}] = [G_x : G_x \cap G_y] < \infty$$

but $gG_xg^{-1} = G_{gx}$ and $gG_yg^{-1} = G_{gy}$ so we have $[G_{gx} : G_{gx} \cap G_{gy}] < \infty$ and $gx \sim gy$. □

Theorem 6.2 *Suppose (G, X) is a measurable group action and G an infinite group, which acts transitively, faithfully, and definably primitively on X. Let $B \triangleleft G$ be the subgroup of finite conjugacy classes. Then either* 1. *or* 2. *holds:*

1. $\dim(G) = \dim(X)$. *In this case either* (a) *or* (b) *holds:*

   (a) *B is non-trivial. Then B is a definable divisible torsion-free Abelian subgroup of G, which has finite index in G, and which acts regularly on X. Also, B is a minimal definable normal subgroup of G.*

   (b) *B is trivial. Then there is $H \triangleleft G$ where H is the unique minimal definable subgroup of G. It is of finite index in G and $H = T^n$, where T is a simple group and n is some positive integer.*

2. $\dim(G) > \dim(X)$. *Then B is trivial, and G acts primitively on X.*

Proof. Consider first the case $\dim(G) = \dim(X)$. First we observe that point-stabilizers G_x are finite by the Orbit-Stabilizer Theorem.

(a) By measurability B is definable, and by Remark 2.2, $B \triangleleft G$. As the action is transitive and faithful, B is not contained in any G_x. So $G = B \cdot G_x$, and thus B acts transitively on X. It also follows that B is of finite index in G, bounded by $|G_x|$. Consider the derived subgroup B'. By Remark 2.3, B' is a finite normal subgroup of G, and so it is trivial. Thus B is Abelian. Therefore B acts regularly on X, and hence is minimal.

It follows that B is either torsion-free or an elementary p-group for some p. But then picking any $b \in B \setminus \{0\}$, since b^G is finite, we may define the normal subgroup $\langle b^G \rangle$ of G.

(b) Suppose N is a non-trivial, definable normal subgroup of G. Again $G = N \cdot G_x$, and thus all definable normal subgroups N of G are of finite index (bounded by $|G_x|$) in G. Thus there can be at most one minimal definable normal subgroup H. As the conjugacy classes here are infinite, Lemma 5.2 implies that minimal normal subgroups exist and are definable. Thus H must exist as the unique one such.

Of course H is also the socle of G, so we may apply Proposition 5.5. Since H is not of forms (a) or (b) it must be a product of definable, isomorphic simple groups. Thus $H = T^n$.

We move on to the case $\dim(G) > \dim(X)$. In this case, point-stabilizers G_x are infinite. We begin by recalling the fundamental equivalence relation on X:

$$x \sim y :\Leftrightarrow [G_x : G_x \cap G_y] < \infty.$$

By Proposition 6.1, $\sim$ is a definable equivalence relation. By the definable primitivity of (G, X), it follows that it has exactly one class or all its classes are trivial.

Claim 6.2.1 *All the classes of $\sim$ are trivial (i. e. of size 1).*

P r o o f. Suppose not, so there is exactly one class. Then all point-stabilizers G_x are commensurable and over all $x, y \in X$, there exists a finite upper bound for the index $[G_x : G_x \cap G_y]$. Thus the Bergman-Lenstra Theorem (see for instance [13, Theorem 4.2.4]) produces a definable normal subgroup $N \triangleleft G$ uniformly commensurable to all the G_x. Since the G_x are infinite N is non-trivial. Since it is normal N must act transitively on X. But it is commensurable with any G_x and so the orbit of any x under N must be finite. But G is infinite, and the G action is assumed faithful and so X must be infinite. This is a contradiction.

We deduce that all the $\sim$ -classes are trivial. □ (Claim 6.2.1)

Claim 6.2.2 *Suppose W is a subgroup of G but is not necessarily definable. Say $W \geq H, K$, where H and K are definable groups with $m = \dim(H) = \dim(K)$ and $\dim(H \cap K) < m$. Then W contains a definable subgroup S, where $\dim(S) > m$, and $|H : H \cap S|, |K : K \cap S| < \infty$.*

P r o o f. Consider $\langle H, K\rangle$: by Remark 3.5 there is a definable subgroup S with $S \leq \langle H, K\rangle \leq W$ and such that $\mathrm{Cos}(H : S \cap H)$ and $\mathrm{Cos}(K : S \cap K)$ are finite sets. Thus S has dimension at least m. If $\dim(S) = m$, then it follows easily that H and K are commensurable. They are not, so $\dim(S) > m$. □ (Claim 6.2.2)

Now we suppose for a contradiction that $x \in X$ and that G_x is not a maximal subgroup of G. So there exists W where $G_x < W < G$. Of course W is not definable. Consider subgroups H^* of G such that

1. H^* is a definable subgroup of W,
2. $\mathrm{Cos}(G_x : G_x \cap H^*)$ is a finite set.

Let H be a subgroup of G of maximal dimension with these properties.

Claim 6.2.3 *If $g \in G_x$, then H and gHg^{-1} are commensurable.*

P r o o f. Suppose there was $g \in G_x$ such that $\dim(H \cap gHg^{-1}) < \dim(H)$. Then by Claim 6.2.2, W would contain some definable S with $\dim(S) > \dim(H)$ and $\mathrm{Cos}(H : S \cap H)$ a finite set.

But $[G_x : H \cap G_x] < \infty$ by assumption, and also

$$[H \cap G_x : H \cap G_x \cap S] \leq [H : H \cap S] < \infty.$$

So we must have $[G_x : H \cap G_x \cap S] < \infty$. Thus $[G_x : S \cap G_x] < \infty$, and so S satisfies both demands for H^* groups. But $\dim(S) > \dim(H)$ and this is a contradiction. □ (Claim 6.2.3)

Firstly we conclude that $\dim(H) > \dim(G_x)$. This is because the fundamental equivalence relation $\sim$ has all trivial classes. So picking $a \in W \setminus G_x$, say $y = a \cdot x$, then $G_y = aG_xa^{-1}$ has $\dim(G_x \cap G_y) < \dim(G_x)$. Then by Claim 6.2.2, we find an H^*-type group S with $\dim(S) > \dim(G_x)$.

Suppose now that $g_1, g_2 \in G_x$, and $g_1(G_x \cap H) = g_2(G_x \cap H)$. Then $g_1^{-1}g_2 \in H$, so $g_1Hg_1^{-1} = g_2Hg_2^{-1}$. Since $[G_x : G_x \cap H] < \infty$, it follows that H has only finitely many images under conjugation by elements of G_x. Claim 6.2.3 shows that all those images are commensurable. So letting

$$H_0 = \bigcap_{g \in G_x} gHg^{-1},$$

it follows that $\dim(H_0) = \dim(H) > \dim(G_x)$. Furthermore H_0 is normalised by G_x, $H_0 \subseteq H \subseteq W$, and H_0 is definable. But now let

$$H_1 = \langle G_x, H_0\rangle = \{gh \mid g \in G_x, h \in H_0\}.$$

Then H_1 is definable and $G_x \subset H_1 \subset W$, and this contradicts the definable maximality of G_x.

Finally we need to show that B is trivial. Suppose for a contradiction that there is a non-trivial finite conjugacy class. Again B is a definable and a characteristic subgroup. B must act transitively on X. Now $B' \lhd G$ is a finite group and so by primitivity and faithfulness, it must be the identity. So B is Abelian and acts transitively and faithfully on X. So B acts regularly.

Fix $x \in X$. It is straightforward that the action of G_x on X is isomorphic to that of G_x on B by conjugation: $g(x^b) \longmapsto g^{-1}bg$. (In fact $G = B \rtimes G_x$.)

Let $b \in B \setminus \{0\}$ and $x \in X$. Then $|\mathrm{Orb}_{G_x}(x^b)| = |b^{G_x}|$, and this is finite. So by the Orbit-Stabilizer Theorem, $|G_x : G_x \cap G_{x^b}| = |b^{G_x}|$ is finite, contradicting $\sim$ having trivial classes. □

We end with a result that ties asymptotic group actions to the theory of finite primitive groups: a very well-developed body of research (see for instance [2]). We first comment that it is straightforward by Łos' Theorem that if $(G, X) = \prod_{i\in I}(G_i, X_i)/\mathcal{U}$ is a non-principal ultraproduct of finite group actions, then it is transitive (faithful) if and only if there exists $U \in \mathcal{U}$ where for each $i \in U$ the component (G_i, X_i) is transitive (faithful).

Proposition 6.3 *Let $(G, X) = \prod_{i\in I}(G_i, X_i)/\mathcal{U}$ be a non-principal ultraproduct of finite group actions. Suppose that the action is transitive, faithful, and $\dim(G) > \dim(X)$. Then the following hold:*

1. *(G, X) is a primitive group action if and only if for all $J \in \mathcal{U}$ there is $j \in J$ where the action (G_j, X_j) is primitive.*

2. *Suppose that (G, X) is primitive. Let $S(x)$ be a formula defining the socle of G (such exists by Lemma 5.3). Then there is $J \in \mathcal{U}$ where for all $j \in J$ the formula $S(x)$ defines the socle in G_j.*

Proof.

1. We start with the left to right direction. Suppose there is $J \in \mathcal{U}$, and for each $j \in J$, (G_j, X_j) is imprimitive. Select B_j a block of imprimitivity for each such $j \in J$. For $j \in I \setminus J$ let $B_j = X_j$. Now let

$$B = \prod_{i\in I} B_i/\mathcal{U}.$$

It is enough to show that B is a block of imprimitivity in (G, X).

Let $\tilde{g} \in G$, where $\tilde{g} = (g_i \mid i \in I)$, let

$$K = \{j \in J \mid g_jB_j \cap B_j = \emptyset\} \quad \text{and} \quad Q = \{j \in J \mid g_jB_j \cap B_j = B_j\}.$$

Since for each $j \in J$ the set B_j is a block of imprimitivity, we have $K \cup Q = J$, and so either $K \in \mathcal{U}$ or $Q \in \mathcal{U}$.

Suppose $K \in \mathcal{U}$. Then $\tilde{g}B \cap B = \emptyset$. For suppose

$$\tilde{x} = (x_i \mid i \in I) \quad \text{and} \quad \tilde{x} \in B \cap \tilde{g}B.$$

Then there is $\tilde{y} = (y_i \mid i \in I)$ such that $\tilde{y} \in B$ and $\tilde{g}\tilde{y} = \tilde{x}$. By Łos' Theorem there is $J' \in \mathcal{U}$ where for $j \in J'$ we have $g_jy_j = x_j$ where $y_j \in B_j$ and $x_j \in B_j$. As $K \cap J' \neq \emptyset$, this is a contradiction.

A similar argument shows that if $Q \in \mathcal{U}$, then $\tilde{g}B = B$. Thus (G, X) is an imprimitive group action.

Now for the right to left implication, suppose that (G, X) is an imprimitive group action. Then by Theorem 6.2, (G, X) is definably imprimitive. So pick $\tilde{x} = (x_i \mid i \in I)$, $\tilde{x} \in X$. Then there are formulae $G_{\tilde{x}}(y)$, $H(y)$ in one free variable such that $G_{\tilde{x}}(y)$ defines the point-stabilizer $G_{\tilde{x}}$ in G and $H(y)$ defines a group H with $G_{\tilde{x}} \subset H \subset G$.

It is a simple verification using Łos' Theorem that there is a set $J \subseteq I$ with $J \in \mathcal{U}$ such that for each $j \in J$ the formula $G_{\tilde{x}}(y)$ defines the point-stabilizer G_{x_j} in G_j, and $H(y)$ defines a group H_j with $G_{x_j} \subset H_j \subset G_j$.

Thus (G_j, X_j) is imprimitive for all $j \in J$.

2. Firstly consider a minimal normal subgroup M of G. We know from Lemma 5.2 and Theorem 6.2 that M exists, is infinite, and is definable by a formula $M(x, \tilde{m})$, where $\tilde{m} = (m_i \mid i \in I)$, say.

By Remark 3.5 for any $x \in M$ there is a natural number n_x such that any element of M is expressible as a product of n_x elements from $x^G \cup (x^{-1})^G$. We may take n_x to be the smallest such natural number. By compactness the n_x have a finite upper bound n. Now writing $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, we find that G satisfies the following sentence:

$$\forall x(M(x, \tilde{m}) \Rightarrow [\forall y(M(y, \tilde{m}) \Rightarrow (\exists x_1x_2 \dots x_n \in x^G)(\textstyle\bigvee_{\bar{\varepsilon}\in\{-1,1\}^n} x_1^{\varepsilon_1}x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} = y))]).$$

Then by Łos' Theorem there is an ultrafilter set $J \subseteq I$ such that for all $j \in J$ we have that $M(G_j, m_j)$ is a normal subgroup of G_j, and

$$G_j \vDash \forall x(M(x, m_j) \Rightarrow [\forall y(M(y, m_j) \Rightarrow (\exists x_1 x_2 \dots x_n \in x^G)(\bigvee_{\bar{\varepsilon} \in \{-1,1\}^n} x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n} = y))]).$$

But this sentence asserts that for any $x \in M(G_j, m_j)$, the set x^G generates $M(G_j, m_j)$. Hence $M(G_j, m_j)$ is a minimal normal subgroup of G_j.

Now by part 1. above we know that there is $J' \subseteq J$ where $J' \in \mathcal{U}$ and for each $j \in J'$ the action (G_j, X_j) is primitive.

But now we appeal to a theorem about finite primitive permutation groups. [2, Lemma 4.3B] says that for a finite primitive permutation group G^*,

(i) either the socle is a minimal normal,

(ii) or the socle is the product of a minimal normal H and $C_{G^*}(H)$.

Suppose that M is the minimal normal defined by $M(x, \tilde{m})$. Now $C_G(M)$ is obviously definable by a formula $C(x, \tilde{m})$, say, and there is a set $J'' \subseteq I$ where $J'' \in \mathcal{U}$ such that for all $j \in J$ the formula $C(x, m_j)$ inteprets the centraliser in G_j of the group defined by $M(x, m_j)$. Thus either the formula $M(x, \tilde{m})$ or the formula

$$(\exists y, z)(M(y, \tilde{m}) \wedge C(z, \tilde{m}) \wedge x = yz)$$

must define the socle on some $J''' \subseteq I$, where $J''' \in \mathcal{U}$. □

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