**Indecomposability Theorem (IT)** Let $G$ be a group definable in a supersimple structure of finite $SU-rank$. Let $\{X_i : i \in I\}$ be a collection of definable subsets of $G$. Then there exists a definable subgroup $H \le \langle X_i : i \in I \rangle \le G$ such that: (i) every element of $H$ is a product of a bounded finite number of elements of the $X_i$'s and their inverses (ii) $X_i/H$ is finite for each $i$.
If $(X_i)^g = X_i$ for every $i \in I$ and $g \in G$, then $H$ can be chosen to be normal in $G$.

**Fact 1** (i) If $G$ is a group of finite $SU-rank$ and $H \le G$ is a definable subgroup of infinite index, then $SU(H) < SU(G)$.
(ii) If $H,K$ are definable then $SU(H \times K) \ge SU(H) + SU(K)$.

**Fact 2** If $G$ is a $BFC-group$, i.e. $\exists n \in \omega \forall g \in G\, |g^G| \le n$, then $G'$ is finite.

## Minimal normal subgroups and the socle

Let $G$ be a group definable in a supersimple structure of finite $SU-rank$. (So $SU = D = S_1$)

**Definition** A minimal (definable) normal subgroup of a group $G$ is a nontrivial proper (definable) normal subgroup of $G$ containing no other (definable) normal subgroup of $G$. The socle $soc(G)$ of $G$ is the subgroup of $G$ generated by all minimal normal subgroups of $G$.

**Lemma 1**
If $G$ has no finite conjugacy classes, then minimal normal subgroups of $G$ exist and are definable.

**Proof** Let $H$ be a minimal normal subgroup of $G$, and take $x \in H \setminus \{e\}$. By IT applied to the infinite set $x^G$, there is an infinite definable $N \le H$ normal in $G$. By minimality of $H$, $H = N$ is definable.
Existence: Let $W$ be a definable normal subgroup of $G$ of minimal rank.
Suppose $W = W_0 > W_1 > \dots$ is a descending chain of normal subgroups of $G$ all definable over some $A$.
It is enough to show it stabilises. As $SU(W_i) = SU(W)$, we have $[W : W_i] < \omega$ for every $i < \omega$.
Hence $W_i \ge W^0_A = \cap\{Z : Z \le W, [W : Z] < \omega, Z$ definable over $A\}$. Note $W^0_A$ is infinite as $W$ is.
Also, $W^0_A$ is normal in $G$, so for $\gamma \in W^0_A \setminus \{e\}$ we have $\langle \gamma^G \rangle \le W^0_A$. Note $\gamma^G$ is infinite.
By IT applied to $\gamma^G$ there is an infinite definable $N \le \langle \gamma^G \rangle$ normal in $G$. If $W_0 > W_1 > \dots$ does not stabilise, then $[W : W^0_A] = \infty$, so $[W : N] = \infty$, so $SU(N) < SU(W)$, a contradiction to the choice of $W$.

**Lemma 2** 1. Any two distinct minimal normal subgroups of any group centralise each other.
2. If $G$ has no nontrivial finite conjugacy classes, then the socle of $G$ is definable and is a finite direct product of minimal normal groups.

**Proof** 1. If $H$ and $K$ are distinct minimal normal sugroups of a group $G$, then $[H,K]$ is normal in $G$ and $[H,K] \subseteq H \cap K$ $([h,k] = h^{-1}h^k \in H$ and $[h,k] = k^{-1}k^h \in K.)$ But $H \cap K = \{e\}$ by minimality.
2. Inductively, if $H_1,\dots,H_n$ are minimal normal subgroups of $G$, then $\langle H_i : i \le n \rangle = H_{i_1} \times \dots \times H_{i_l}$ for some $i_1 < \dots < i_l \le n$: indeed, if this holds and $H_{n+1}$ is another minimal normal subgroup, then $\langle H_i : i \le n \rangle \cap H_{n+1}$ is either trivial or equal to $H_{n+1}$. In the former case $\langle H_i : i \le n+1 \rangle = H_{i_1} \times \dots \times H_{i_l} \times H_{n+1}$, and in the latter case $\langle H_i : i \le n+1 \rangle = H_{i_1} \times \dots \times H_{i_l}$.
As $G$ has no nontrivial finite conjugacy classes, every minimal normal subgroup of $G$ is infinite and definable by Lemma 1. As $SU(H_1 \times \dots \times H_l) \ge l$ for infnite $H_1,\dots,H_l$, we conclude that $Soc(G) = H_1 \times \dots \times H_l$ for some minimal normal subgroups $H_1 \times \dots \times H_l$ of $G$.

**Fact 3** If $A$ is an Abelian group with no nontrivial proper definable characteristic subgroups, then either $A$ is an elementary $p-group$ for some prime $p$, or $A$ is $torsion-free$ and divisible.

**Proposition 1** Suppose that $G$ eliminates $\exists^\infty$. Let $M$ be a minimal definable normal subgroup of $G$.
If $M$ is infinite, then one of the following holds:
1) $M$ is an elementary $p-group$. 2) $M$ is a $\mathbb{Q}-vector$ space. 3) $M$ is a minimal normal subgroup of $G$, and is a finite direct product of isomorphic, definable, simple groups.

**Proof** Put $B := \{a \in M : |a^G| < \omega\}$. $B$ is a definable normal subgroup of $G$ (as $G$ eliminates $\exists^\infty$),
So $B = \{e\}$ or $B = M$. If $B = M$, then, by Fact 2, $M' =$ is finite (so definable), and normal in $G$ (as it is characteristic in $M$ and $M$ is normal in $G$), so $M' = \{e\}$, i.e. $M$ is abelian.
As $M$ has no definable characteristic proper nontrivial subgroups, $M$ satisfies 1) or 2) by Fact 3.
If $B = \{e\}$, then by Lemma 1 there is a minimal normal subgroup $T$ of $M$, and $T$ is definable.
$T^x$ is also a minimal normal subgroup of $M$ for any $x \in G$, so $T^x \cap T^y = \{e\}$ or $T^x = T^y$ whenever $x,y \in G$.
Thus $\Omega := \{T^x : x \in G\}$ is a family of pairwise commuting minimal normal subgroups of $M$.
As in the proof of Lemma 2 we get that $T_0 := \langle \cup \Omega \rangle = T^{x_1} \times \dots \times T^{x_n}$ for some $x_1,\dots,x_n \in G$.
Hence $T_0$ is definable, normal in $G$ and contained in $M$, so $T_0 = M$ by minimality of $M$.
It remains to show that $T$ is simple. If $S$ is a normal subgroup of $T$, then, as $M = T_0 = T \times Y$ for some $Y \le M$, we get that $S$ is normal in $M$, By minimality of $T$, $S = T$ or $S = \{e\}$.

## Measurable group actions

Recall a structure $M$ is measurable if there is a function $h = (dim,\mu) : Def(M) \to \mathbb{N} \times \mathbb{R}^{>0} \cup \{(0,0)\}$ which assumes finitely many values on definable families of sets, $h(X) = (0,|X|)$ for finite $X$, $h$ is $\varnothing-definable$, and satisfies "Fubini Property". In particular, if $d = dim(X) = dim(Y)$ then $dim(X \cup Y) = d$ and if $X \cap Y = \varnothing$ then $\mu(X \cup Y) = \mu(X) + \mu(Y)$.

Let $(G,X)$ be a transitive action of a group $G$ on a set $X$, all definable in some measurable structure.
If $x \in X$ and $Y \subseteq X^n$, put $G_x := \{g \in G : gx = x\}$, $Gx = \{gx : g \in G\}$ and $G_Y = \{g \in G : g[Y] = Y\}$.
Note $G_{gx} = gG_xg^{-1} = G_x^{g^{-1}}$.
If $K,H \le G$, write $K \lesssim H$ if $[K : K \cap H] < \omega$, and $K \approx H$ if $K \lesssim H \lesssim K$ (commensurability)

**Definition** The action $(G,X)$ is [definably] primitive if there is no proper nontrivial [definable] equivalence relation $E$ on $X$ preserved by $G$ (i.e. $\forall x,y \in X \forall g \in G\, E(x,y) \Rightarrow E(gx,gy)$).
Equivalently, for any $x \in X$ there is no [definable] group $H$ with $G_x < H < G$ (proper inlucsions).

**Proposition 2** Let $G$ be a measurable group acting definably and transitively on a definable set $X$.
Define $\sim$ on $X$ by $x \sim y \Leftrightarrow G_x \lesssim G_y$. Then $\sim$ is a definable $G-invariant$ equivalence relation.

**Proof** As $G$ is measurable, there are only finitely many possibilities on $\left[G_x : G_x \cap G_y\right] < \omega$ (as there are finitely many possiblities on $\mu(G_x \cap G_y)$, and $\mu\left(G_x\right) = \left[G_x : G_x \cap G_y\right]\mu\left(G_x \cap G_y\right)$
Thus $\sim$ is definable. Transitivity of $\sim$ follows from transitivity of $\approx$.
$\sim$ is $G-invariant$, as $\left[G_x : G_x \cap G_y\right] = \left[(G_x)^{g^{-1}} : (G_x)^{g^{-1}} \cap (G_y)^{g^{-1}}\right] = \left[G_{gx} : G_{gx} \cap G_{gy}\right]$ for $x,y \in X, g \in G$.
Symmetry: As $dim\left(G_x\right) = dim\left((G_x)^{g^{-1}}\right) = dim\left(G_{gx}\right)$ for any $g \in G$ and $x \in X$, we get by transitivity of the action that $dim\left(G_x\right) = dim(G_y)$ for any $x,y \in X$. Now if $x \sim y$, then $\left[G_x : G_x \cap G_y\right] < \omega$ so $dim\left(G_x \cap G_y\right) = dim\left(G_x\right) = dim(G_y)$, hence $\left[G_y : G_x \cap G_y\right] < \omega$, hence $y \sim x$.

**Theorem** Suppose $(G,X)$ is a measurable group action and $G$ an infinite group, which acts transitively, faithfully, and definably primitively on $X$. Let $B = \left\{g \in G : |g^G| < \omega\right\}$. Then:
1a) If $dim(G) = dim(X)$ and $B \ne \{e\}$, then $B$ is a definable divisible $torsion-free$ Abelian subgroup of $G$ of finite index and acts regularly on $X$. Also, $B$ is a minimal normal subgroup of $G$.
1b) If $dim(G) = dim(X)$ and $B = \{e\}$, then there is a unique minimal definable normal subgroup $H$ of $G$. $[G : H] < \omega$, and $H = T^n$ for a simple group $T$ and $n \in \omega$.
2) If $Dim(G) > dim(X)$, then $B = \{e\}$ and $G$ acts primitively on $X$.

**Proof** Note that by definable primitivity, if $N$ is a nontrivial definable normal subgroup of $G$ then $N$ acts transitively on $X$, as otherwise $E(x,y)$ given by $Nx = Ny$ is a definable nontrivial equivalence relation on $X$ invariant under the action of $G$ (by normality of $N$).
Assume first that $dim(X) = dim(G)$. Then $\forall x \in X\, |G_x| < \omega$ by Fubini applied to $G \ni g \mapsto gx \in X$.
Case 1a): $B \ne \{e\}$. By measurability, $B$ is a definable normal subgroup of $G$, so it acts transitively on $X$, so $G = BG_x$ and $[G : B] \le |G_x| < \omega$. As $B$ is $BFC$, $B'$ is a finite normal subgroup of $G$. It cannot act transitively on $X$, so $B' = \{e\}$, i.e. $B$ is abelian, hence divisible $torsion-free$ or or an elementary $p-group$. By abelianity, $B$ acts regularly on $X$ ($b(gx) = g(bx)$ is determined by $bx$).
If $B$ is an elementary $p-group$ then for any $b \in B \setminus \{e\}$ we have a finite normal subgroup $\langle b^G \rangle$ of $B$, so it cannot act transitively on $X$, a contradiction. So $B$ is divisible and $torsion-free$.
Case 1b) $B = \{e\}$. As above, if $N$ is a nontrivial definable normal subgroup, then $G = NG_x$, so $[G : N] \le |G_x| < \omega$. Thus there can be at most one minimal definable normal subgroup of $G$.
But by Lemma 1 there is a minimal normal subgroup $H$ of $G$ and it is definable; it must be unique.
By Proposition 1, $H$ is a product of finitely many isomorhpic simple groups.

Case 2: $dim(G) > dim(X)$. Then $G_x$ is infinite for every $x \in X$. Recall $x \sim y \Leftrightarrow G_x \approx G_y$ is a definable $G-invariant$ equivalence relation, so, by definable primitivity, it has only one class or all its classes are trivial. If there is only one $\sim -classs$, i.e. all $G_x$, $x \in X$ are commensurable, then, as $(G_x)^g = G_{g^{-1}x}$ for all $g \in G$, $x \in X$, Schlichting's theorem yields a definable normal sugroup $N$ of $G$ commensurable with $G_x$ for every $x \in X$. In particular, $N$ is infinite so nontivial, so it acts transitively on $X$. But $Nx$ is finite as $N/G_x$ is finite, a contradiction. So all $\sim -classes$ have size 1.

**Claim 1** If $W \le G$ (not necessarily definable), $H,K \le W$ are definable with $m = dim(H) = dim(K)$ and $dim(H \cap K) < m$, then $W$ contains a definable subgroup $S$ with $dim(S) > m$, and $H,K \lesssim S$.
pf. By IT there is a definable $S \le \langle K,H \rangle \le W$ with $H,K \lesssim S$. If $dim(S) = m$ then $K \approx H$, so we must have $dim(S) > m$.

Now suppose for a contradiction that there is $x \in X$ such that $G_x$ is not maximal, so there is $W$ with $G_x < W < G$. Let $H \le G$ be of maximal possible dimension satisfying:
i) $H$ is a definable subgroup of $W$ ii) $G_x \lesssim H$.

**Claim 2** If $g \in G_x$, then $H \approx H^g$.
pf. If not, then $dim\left(H \cap H^g\right) < dim(H)$. Then by Claim 1 there is a definable $S \le W$ with $dim(S) > dim(H)$ and $H \lesssim S$, so $G_x \lesssim H \lesssim S$, so $S$ satifies i) and ii), a contradiction.

Pick $a \in W \setminus G_x$. Then $y := ax \ne x$, and $dim\left(G_x \cap G_y\right) < dim(G_x)$ as $G_x$ is not commensurable with $G_y$.
Then by Claim 1 there is $S$ satisfying i), ii) with $dim(S) > dim(G_x)$. Hence $dim(H) > dim(G_x)$.
Now if $g_1,g_2 \in G_x$ and $g_1\left(G_x \cap H\right) = g_2(G_x \cap H)$, then $g_2^{-1}g_1 \in H$, so $g_2^{-1}g_1H = Hg_2^{-1}g_1$ so $g_1Hg_1^{-1} = g_2Hg_2^{-1}$. As $\left[G_x : G_x \cap H\right] < \omega$ and conjugates of $H$ by elements of $G_x$ are commensurable it follows that for $H_0 := \cap_{g \in G_x} H^g$ we have $dim(H_0) = dim(H) > dim(G_x)$. As $H_0$ is normalised by $G_x$, $H_1 := \langle G_x,H_0 \rangle = \left\{gh : g \in G_x, h \in H_0\right\}$ is definable. As $H_0 \le H \le W$ and $G_x \le W$ we get $G_x \le H_1 \le W$, and $G_x \ne H_1$ as $dim(H_1) \ge dim\left(H_0\right) = dim(H) > dim(G_x)$. This contradicts definable maximality of $G_x$ in $G$.
It is left to show that $B = \{e\}$. As before, $B'$ is finite and if it were nontrivial then it would act transitively on $X$, a contradiction. So $B' = \{e\}$ and $B$ is abelian. As $B$ acts transitively faithfully on $X$, it acts regularly.
Let $b \in B \setminus \{e\}$ and $x \in X$. Then $bx \ne x$ and for any $g \in G_x$ we have $g(bx) = \left(gbg^{-1}\right)gx = b^{g^{-1}}x \in b^G \cdot x$
so $G_x \cdot b$ is finite. Hence $\left[G_x : G_x \cap G_b\right] < \omega$, contradicting $\sim$ having only trivial classes.

**Proposition 3** Let $(G,X) = \prod_{i \in I}(G_i,X_i)/U$ be a (measurable) nonprincipal ultraproduct of finite group actions. Suppose the action is transitive, faithful, and $dim(G) > dim(X)$. Then:
1. $(G,X)$ is a primitive group action iff for all $J \in U$ there is $j \in J$ such that $(G_j, X_j)$ is primitive
2. Suppose $(G,X)$ is primitive.
Then there is $J \in U$ and a formula $S(x,\bar{m})$ such that $S\left(x,m_j\right)$ defines $soc\left(G_j\right)$ for all $j \in J$.

**Sketch of a proof** 1. Left to right: If $J \in U$ and $(G_j,X_j)$ is imprimitive for every $j \in J$, witnessed by a block (nontrivial equivalence class of a $G-invariant$ equivalence relation) $B_j$, then $\prod B_j/U$ is a block for $(G,X)$ (by Łoś' Theorem).
Right to left: If $(G,X)$ is imprimitive, then by the theorem above it is definably imprimitive, witnessed by some $\bar{x} \in X$ and definable $H = \phi(G)$ with $G_{\bar{x}} < H < G$. Applying Łoś' Theorem, we obtain the same on $U-many$ coordinates.
2. Consider a minimal normal subgroup $M$ of $G$. By Lemma 1(when $B = \{e\}$) and the theorem ($M := B$ when $B \ne \{e\}$) $M$ exists and is definable by a formula $\phi(x,\bar{m})$.
By compactness there is $n < \omega$ such that for every $x \in X$, $M = \left\{x_1 \dots x_n : x_i \in X \cup X^{-1}\right\}$, which can be expressed by a sentence $\psi(\bar{m})$. By Łoś' Theorem, $\phi(G_j)$ is a normal subgroup of $G_j$ and $G_j \vDash \psi(m_j)$ for $U-many$ $j's$, so $\phi(G_j,m_j)$ is a minimal normal subgroup of $G_j$.
Also, by Part 1, $\left(G_j,X_j\right)$ is primitive for $U-many$ $j's$. Now we will use:

> Fact: (if $G^*,X^*$) is finite primitve action, then either $soc(G^*)$ is minimal normal, or $soc(G^*)$ is a product of a minimal normal $H$ and $C_G(H)$.

One of these two possibilities holds for $U-many$ $j's$; in the first case $\phi(x,m_j)$ defines the $soc(G_j)$ and in the second case,
the formula defining the product of $\phi(G_j,m_j)$ with the centraliser of $\phi\left(G_j,m_j\right)$ in $G_j$ defines $soc(G_j)$ for $U-many$ $j's$.