PERMUTATION GROUPS IN o-MINIMAL STRUCTURES

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1. *Introduction*

In this paper we develop a structure theory for transitive permutation groups definable in o-minimal structures. We fix an o-minimal structure $\mathscr{M}$, a group G definable in $\mathscr{M}$, and a set Ω and a faithful transitive action of G on Ω definable in $\mathscr{M}$, and talk of the *permutation group* (G, Ω). Often, we are concerned with *definably primitive* permutation groups (G, Ω); this means that there is no proper non-trivial definable G-invariant equivalence relation on Ω, so definable primitivity is equivalent to a point stabiliser G_α being a maximal definable subgroup of G. Of course, since any group definable in an o-minimal structure has the descending chain condition on definable subgroups [**23**] we expect many questions on definable transitive permutation groups to reduce to questions on definably primitive ones.

Recall that a group G definable in an o-minimal structure is said to be *connected* if there is no proper definable subgroup of finite index. In some places, if G is a group definable in $\mathscr{M}$ we must distinguish between definability in the full ambient structure $\mathscr{M}$ and **G**-definability, which means definability in the pure group $\mathbf{G} := (G, .)$; for example, *G is* **G**-*definably connected* means that G does not contain proper subgroups of finite index which are definable in the group structure. By *definable*, we always mean definability in $\mathscr{M}$. In some situations, when there is a field R definable in $\mathscr{M}$, we say a set is *R-semialgebraic*, meaning that it is definable in $(R, +, .)$. We call a permutation group (G, Ω) *R-semialgebraic* if G, Ω and the action of G on Ω can all be defined in the pure field structure of a real closed field R. If R is clear from the context, we also just write 'semialgebraic'.

Our main theorem is the following.

THEOREM 1.1. *Let* (G, Ω) *be a definably primitive infinite permutation group definable in an o-minimal structure. In the case when G has a non-trivial abelian normal subgroup, assume also that G is connected and not regular. Then there is a definable real closed field R and an R-semialgebraic permutation group* (G^*, Ω^*) *such that* (G, Ω) *and* (G^*, Ω^*) *are definably isomorphic as permutation groups.*

To see that in this theorem we need an extra hypothesis when there is an infinite abelian normal subgroup, consider the additive group of **Q** acting regularly on itself. Since **Q** is definably simple, this is a definably primitive *regular* action, but there is no definable field. Clearly, this example is not primitive; see the end of Section 4, and the problems at the end of the paper, for more on this.

Theorem 1.1 is deduced from the following theorem, which gives a fine structure theory for definably primitive permutation groups and is an analogue of the O'Nan–Scott theorem for finite primitive permutation groups. (For the latter, see [**28**], or [**10**] for a full proof.) Theorem 1.2 is a tool for reducing questions about

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definably primitive permutation groups to questions in representation theory (case 1 below), and to questions about definably maximal subgroups of groups which are close to being simple (case 2 below).

If G is a group definable in an o-minimal structure, then by results in [**23**, Section 2], G has the descending chain condition on definable subgroups, so any definable non-trivial normal subgroup contains a minimal definable (non-trivial) normal subgroup. Also, the minimal definable normal subgroups commute. We let the *definable socle* of G, denoted $\mathrm{Soc}_d(G)$, be the subgroup generated by the minimal definable normal subgroups of G. We will show in Section 3 below that if G acts definably primitively on Ω, then either G has a unique minimal definable normal subgroup N (acting regularly if N is abelian), or there are exactly two minimal definable normal subgroups T_1 and T_2 definably isomorphic to each other, definably simple and each acting regularly on Ω. Thus, in the first case $\mathrm{Soc}_d(G) = N$ and in the second case $\mathrm{Soc}_d(G) = T_1 \times T_2$, so in both cases $\mathrm{Soc}_d(G)$ is itself a definable subgroup.

Before we state the theorem, note that if G contains a regular subgroup, then Ω can be identified with this subgroup (definably, if the subgroup is definable). If G does not contain a regular subgroup, which happens only in the case where G has a unique minimal normal subgroup, then Ω is given implicitly as the right coset space G/G_α, where G_α denotes the stabiliser in G of $\alpha \in \Omega$. Thus, in cases 1, 3 and 4 below we describe the action essentially by specifying G_α, but in case 2, the only information available is that G_α is a definably maximal subgroup of G.

THEOREM 1.2. *Let (G, Ω) be a definably primitive permutation group definable in an o-minimal structure $\mathscr{M}$, with Ω infinite. Let N be a minimal non-trivial definable normal subgroup of G, and put $B := \mathrm{Soc}_d(G)$. Then N is connected, and B is definable, and either B is abelian or there is an infinite definable non-abelian definably simple group T and a positive integer k such that $B = T_1 \times \ldots \times T_k$, with each T_i definable and definably isomorphic to T. Furthermore, (G, Ω) is of one of the types described below, where $\alpha \in \Omega$.*

Type 1 (*affine group*): Here $N = B$ is a divisible torsion-free abelian group, and so acts regularly on Ω, and $G = N \rtimes G_\alpha$ (a semi-direct product), and the action of G_α on Ω is the same as its action on N by conjugation (under the natural identification of N with Ω, identifying the identity with α).

Type 2 (*unique non-abelian minimal normal subgroup*): Here $N = B = T$ is definably simple, and there is a definable real closed field R such that N is definably isomorphic to a group which is semialgebraic over R. We have $N \leqslant G \leqslant \mathrm{Aut}(N)$.

Type 3 (*simple diagonal action*): Let

$$W := \{(a_1, \ldots, a_k).\pi : a_i \in \mathrm{Aut}(T), \pi \in S_k, a_i \equiv a_j \bmod \mathrm{Inn}(T) \text{ for all } i, j\},$$

where $\pi \in S_k$ permutes the a_i naturally. Then W has a natural group structure whose socle $\mathrm{Soc}_d(W)$ is the obvious copy of T^k which we identify with B. Now (G, Ω) is said to have *type* 3 if G is a subgroup of the group $W = B.(\mathrm{Out}(T) \times S_k)$ (possibly non-split), Ω can be identified with the coset space W/W_α where

$$W_\alpha := \{(a, \ldots, a)\,\pi : a \in \mathrm{Aut}(T), \pi \in S_k\},$$

with G having the induced action on Ω, and one of the following holds:

(i) $k = 2$;

(ii) G induces a primitive action on $\mathscr{T} := \{T_1, \ldots, T_k\}$.

In addition, in this case, the following will hold: $W_\alpha \cong \mathrm{Aut}(T) \times S_k$ and $B_\alpha \cong T$, and if P denotes the permutation group induced by G on $\mathscr{T}$, then $G_\alpha \leqslant \mathrm{Aut}(T) \times P$ and $G \leqslant B.(\mathrm{Out}(T) \times P)$. Also, in case (i), G has exactly two minimal definable normal subgroups T_1 and T_2 each acting regularly on Ω, and in case (ii), $B = N$ is the unique minimal definable normal subgroup of G.

Type 4 (*product action*): Here, there is a permutation group (H, Γ) of type 2 or 3, and G is a subgroup of a permutation group $H \operatorname{wr} S_\ell$ in its product action, where the action of H in each coordinate is that on Γ.

More precisely, let (H, Γ) be an infinite definably simple non-abelian definably primitive permutation group definable in $\mathscr{M}$, and suppose that (H, Γ) has type 2 or 3. There is $\ell \in \mathbf{N}$ with $\ell > 1$ such that Ω is identified (via a definable bijection) with Γ^ℓ. Let $W = H \operatorname{wr} S_\ell$, acting on Ω in the natural way: elements of H act coordinatewise, and elements $\pi \in S_\ell$ act by the rule $(\alpha_1, \ldots, \alpha_\ell)\pi = (\alpha_{1\pi^{-1}}, \ldots, \alpha_{\ell\pi^{-1}})$ for $(\alpha_1, \ldots, \alpha_\ell) \in \Omega$. From this it follows that if $K := \mathrm{Soc}_d(H)$, then $\mathrm{Soc}_d(W) = K^\ell$, and W has an induced action on these ℓ factors.

We say that (G, Ω) is of *type* 4 if G is a subgroup of W with the induced action on Ω, the definable socle B of G is $\mathrm{Soc}_d(W)$ (so $\mathrm{Soc}_d(W) \leqslant G \leqslant W$ and $B = K^\ell$), G acts transitively on the ℓ factors in K^ℓ, and one of the following holds.

(i) (H, Γ) is of type 2, $K = T$ and $k = \ell$, and $B = N$ is the unique minimal definable normal subgroup of G;

(ii) (H, Γ) is of type 3, $K = T^{k/\ell}$, and G and H both have m minimal definable normal subgroups, with $m \leqslant 2$. If $m = 2$ then each of these minimal definable normal subgroups of G acts regularly on Ω.

From the above description it follows that if $\gamma \in \Gamma$ and $\alpha = (\gamma, \ldots, \gamma) \in \Omega$, then $W_\alpha = H_\gamma \operatorname{wr} S_\ell$, and $B_\alpha = (K_\gamma)^\ell$.

Remark 1.3. (1) In Theorem 1.2, if we assume that G is connected, then the examples are far more restricted. For example, in case (2) we have $G = T$, in case (3) we have $k = 2$ and $G = T^2$ acting on cosets of a diagonal subgroup, and case (4) does not arise.

(2) Except in case (1) of Theorem 1.2, it follows for example from [**19**, Claim 2.8] that $\dim(B) = \dim(G)$ (in the sense to be defined) and hence the connected component G^o of G is equal to B.

(3) The description in case (3) is intricate. A typical example, when T is a definable non-abelian infinite definably simple group and k is a positive integer, would be $T \operatorname{wr} S_k$ acting on right cosets of $D \times S_k$, where D is a diagonal subgroup of T^k isomorphic to T.

The main step from Theorem 1.2 to Theorem 1.1 is provided by the following proposition, which gives quite detailed information (and justifies the terminology) for the affine groups in Theorem 1.2, even when G is not assumed to be connected.

Proposition 1.4. *Assume that (G, Ω) satisfies the assumptions of Theorem 1.2, and has an abelian minimal definable normal subgroup N. Suppose that for $\alpha \in \Omega$ the point stabiliser $L = G_\alpha$ is infinite. Then there is a definable real closed field R such that N has definably the structure of a vector space over R of finite dimension m say, and L*

is definably isomorphic to a subgroup L^* of $\mathrm{GL}(m, R)$*, and the action of L on N by conjugation is isomorphic to the natural induced R-linear action of L^*. Furthermore, if G is connected, then L^* is R-semialgebraic.*

By the *dimension* of a definable set, we always mean the o-minimal dimension, given either by the cell decomposition, or by the algebraic closure operation (and these notions extend to interpretable sets). We apply Theorem 1.1 to get the following description of definable permutation groups (G, Ω) where $\dim(\Omega) = 1$. A permutation group (G, Ω) is k-homogeneous if it is transitive on the set of unordered k-subsets of Ω, and *sharply k-homogeneous* if in addition the setwise stabiliser of any unordered k-set is the identity. The notion of a *cyclic* ordering is defined in Section 6 below.

THEOREM 1.5. *Let (G, Ω) be a definable transitive permutation group in an o-minimal structure M, such that G is connected and* $\dim(\Omega) = 1$*. Then one of the following occurs.*

(1) $\dim(G) = 1$*, G acts regularly on Ω, and G is isomorphic to either $\bigoplus_{\delta \in \Delta} \mathbf{Q}$ or $\bigoplus_{p \in P} \mathbf{Z}_{p^\infty} \oplus \bigoplus_{\delta \in \Delta} \mathbf{Q}$, where P is the set of primes, and Δ is an index set. In the first case, G is an ordered group, and so preserves a linear order on Ω (which is definable), and in the second case, G is cyclically ordered and preserves a definable cyclic order on Ω, but no linear order.*

(2) $\dim(G) = 2$*, and there is a definable real closed field R such that G is definably isomorphic to the* 1*-dimensional affine group $R^+ \rtimes R_{>0}$ in its natural* 2*-homogeneous action on R.*

(3) $\dim(G) = 3$*, and there is a definable real closed field R such that G is isomorphic to a finite cover of* $\mathrm{PSL}(2, R)$ *with cyclic centre, acting as a group of automorphisms of a cyclically ordered set.*

In (3), the proof gives slightly more information. The group $G/Z(G)$, and its induced 2-transitive action on the set of maximal blocks of imprimitivity (which may be singletons) is *definably* isomorphic to the action of $\mathrm{PSL}(2, R)$ on $\mathrm{PG}(1, R)$.

This is an analogue of Hrushovski's classification [**6**] of groups of finite Morley rank acting definably and transitively on a strongly minimal set (where very similar examples arise, but over an algebraically closed field). It is also related to Brouwer's classification of transitive transformation groups on 1-manifolds (which is used in our proof). Theorem 1.5 was first proved in a different way by Mosley in [**13**] (though that proof rests on [**14**], in which elimination of imaginaries is assumed for the ambient structure, together with a version of the open mapping theorem). It is probably not hard to adapt Mosley's proof, replacing his appeal to [**14**] by use of more recent results described in Section 2, and thereby avoiding appeal to Theorem 1.1 or the Brouwer classification.

In Section 2 below we state some results on o-minimality which are used heavily. These mostly come from [**19**] and [**20**] and are also surveyed in [**21**]. Theorem 1.2 is proved in Section 3. It is a routine adaptation of [**11**, Theorem 1.1] (which was proved for definably primitive permutation groups of finite Morley rank and was itself an adaptation of [**10**]). Section 4 contains the core of the paper, a proof that in the abelian socle case with non-trivial connected point stabiliser, the permutation group is semialgebraic. We prove Theorem 1.1 in Section 5. There, the non-abelian socle case is handled by a Lie algebra argument suggested by Peterzil. Theorem 1.5 is proved in Sections 6–8. Finally, we discuss some applications in Section 9.

2. *Some preliminary results*

In this section we list some results on groups in o-minimal structures which are used heavily. Throughout the section, $\mathcal{M}$ is an o-minimal structure, and G is a group definable in $\mathcal{M}$. We denote by G^o the connected component of G, that is, the smallest $\mathcal{M}$-definable subgroup of finite index. If R is a real closed field, then $R(i)$ denotes the algebraically closed extension of degree 2 (so $i^2 = -1$).

THEOREM 2.1 [**18**, Theorem 4.1]. *Let $K = (K, +, .)$ be an infinite ring without zero divisors definable in $\mathcal{M}$. Then K is a division ring and there is a 1-dimensional $\mathcal{M}$-definable subring R of K which is a real closed field such that K is R, its algebraic closure $R(i)$, or the skew field of quaternions $H(R)$ over R.*

Clearly, a real closed field R definable in an o-minimal structure $\mathcal{M}$ cannot admit non-trivial definable automorphisms as the fixed field of an automorphism always contains the prime subfield, contradicting Theorem 2.1 unless the automorphism is the identity.

LEMMA 2.2. *Let R be a definable real closed field, n be a positive integer, and σ be a definable group endomorphism of $V := R^n$. Then σ is R-linear.*

Proof. Let $K := \{a \in R : \text{for all } v \in V(\sigma(av) = a\sigma(v))\}$. Then K is a definable subfield of R such that σ acts K-linearly. Since $\mathbf{Q} \subset K$, by Theorem 2.1, $K = R$.

In particular, in an o-minimal structure any definable R-semilinear transformation of a definable R-vector space must be R-linear.

It follows also that if α is a definable non-trivial automorphism of $R(i)$ then $\text{fix}(\alpha) \cap R = R$, and hence α is complex conjugation. Similarly, any definable automorphism of the skew-field of quaternions $H(R)$ is R-linear. It can be proved much as in the real case that the group of automorphisms of $H(R)$ definable in a given o-minimal structure coincides with the group of inner automorphisms and is definably isomorphic to $SO(3, R)$ (see [**27**, 11.29]). For this one first checks that $SO(3, R)$ acts on $H(R)$ as conjugation by elements of norm 1. Using the fact that any definable automorphism of $H(R)$ must be R-linear, it follows that such an automorphism is orthogonal with respect to the inner product $\langle x \mid y \rangle = \bar{x}y + \bar{y}x$ of $H(R)$. Furthermore, $H(R)$ splits as a direct sum $R \oplus P$, where R is the centre and P is the 3-dimensional R-linear subspace of *pure quaternions* $\{x \in H(R) : x^2 \in R_{\leqslant 0}\}$. It is clear that every automorphism of $H(R)$ preserves P and also the anisotropic quadratic form $x \longmapsto \|x\|^2$ on P, whence $\text{Aut}_R(H(R)) \subseteq O(3, R)$. Thus, we have $SO(3, R) \leqslant \text{Aut}_R(H(R)) \leqslant O(3, R)$ where the index of $SO(3, R)$ in $O(3, R)$ is 2, so the result follows from the fact that conjugation in $H(R)$ is an element of $O(3, R)$, but is an anti-automorphism rather than an automorphism because $H(R)$ is not commutative.

Next, we quote a result which stems from [**23**], but was proved in this generality in [**19**]. For the definition of a $C^{(p)}$-manifold in an o-minimal structure, see [**19**]. The final assertion below follows from the fact that in $(\mathbf{R}, +, .)$, if $U \subseteq \mathbf{R}^n$ is open and $f : U \longrightarrow \mathbf{R}^k$ is definable, then f is analytic on a dense open subset of U.

THEOREM 2.3 [**19**, Theorem 2.11]. *Let (G, Ω) be a definable transitive permutation group, and $p \geqslant 0$ (with $p = 0$ unless $\mathcal{M}$ expands an ordered field).*

(i) *The group G has a definable $C^{(p)}$-manifold structure.*

(ii) *There is a definable $C^{(p)}$-manifold structure on Ω, such that, with G carrying the manifold structure of* (i), *the group action $\Omega \times G \longrightarrow \Omega$ is $C^{(p)}$.*

If $\mathscr{M}$ is the pure field of reals, then the manifold structures and the group action and operation can be taken to be analytic.

A group G is called *unidimensional* if there is a definable chart $\langle U, \varphi, n \rangle$ on G around the identity elements such that $\varphi(U)$ is a product $I_1 \times \ldots \times I_k$ of *transitive* and *pairwise non-orthogonal* intervals $I_1, \ldots, I_k$; that is, for all $a, b \in I_j$ there are open definably homeomorphic subintervals $O_a, O_b \subseteq I_j$ containing a and b respectively, and for all I_i, I_j there are definably homeomorphic subintervals $O_i \subseteq I_i$, $O_j \subseteq I_j$.

THEOREM 2.4 [**19**, Theorem 3.1, Theorem 3.2]. *If G is an infinite definable* **G**-*definably connected centreless group, it is a direct product of unidimensional* **G**-*definable centreless* **G**-*definably connected subgroups. Furthermore, if G is unidimensional, then there is a definable real closed field R and a definable group $H \leqslant \mathrm{GL}(n, R)$ definably isomorphic to G.*

THEOREM 2.5 [**19**, Theorem 4.1]. *Assume that G is infinite,* **G**-*definably connected and has no non-trivial abelian normal subgroup. Then G is the direct product of* **G**-*definable subgroups $H_1, \ldots, H_k$ such that for every $i \in \{1, \ldots, k\}$ there is a definable real closed field R_i and a definable isomorphism between H_i and a semialgebraic subgroup of* $\mathrm{GL}(n, R_i)$. *Every H_i is* $\mathbf{H}_i$-*definably simple and its $\mathscr{M}$-definably connected component H_i^o is definably simple.*

THEOREM 2.6 [**22**, Theorem 4.1]. *Assume that $\mathscr{M}$ expands a real closed field R and that G is a connected definable subgroup of* $\mathrm{GL}(n, R)$. *Then there is a semialgebraic normal subgroup $H \trianglelefteq G$ such that G/H is abelian.*

3. *The O'Nan–Scott reduction*

Proof of Theorem 1.2. We begin with a definably primitive permutation group (G, Ω), definable in an o-minimal structure $\mathscr{M}$, where Ω is infinite. By the descending chain condition on definable subgroups, G has a minimal non-trivial definable normal subgroup N, and N is transitive on Ω (otherwise the N-orbits will be classes of a definable G-congruence on Ω). By its transitivity, N is infinite, so by minimality it is connected. The centraliser $C_G(N)$ is also a definable normal subgroup of G. There are three cases.

(i) $C_G(N) = N$.
(ii) $C_G(N) = 1$.
(iii) $C_G(N) \neq 1$ and $C_G(N) \neq N$.

In cases (i) and (iii), by definable primitivity, $C_G(N)$ is transitive on Ω. Since any two commuting transitive groups are regular, it follows in these cases that N and $C_G(N)$ are both regular. Since G is acting faithfully and any transitive abelian permutation group is regular, case (i) holds if and only if N is abelian. In case (iii), by regularity, $C_G(N)$ is also a minimal definable normal subgroup of G. Furthermore, in this case, the actions of N and $C_G(N)$ are isomorphic to the left and right regular representations of the same group, so $N \cong C_G(N)$. (This isomorphism is definable: fix $\alpha \in \Omega$; then $g \in N$ maps to $g' \in C_G(N)$, where $\alpha^{gg'} = \alpha$.) Also, if M is any minimal definable normal subgroup of G with $M \neq N$, then $M \cap N = 1$, so $M = C_G(N)$. Thus, by (i)–(iii), precisely one of the following cases holds.

(1) $C_G(N) = N$ and N is the unique minimal definable normal subgroup and is abelian and regular.

(2) $C_G(N) = 1$ and N is the unique minimal definable normal subgroup of G and is non-abelian.

(3) $C_G(N)$ and N are the only two minimal definable normal subgroups of G and are definably isomorphic to each other, and non-abelian, and each is regular.

Let B be the definable socle of G. By the previous paragraph we now see that B is itself definable, namely in cases (1) and (2), $B = N$, and in case (3), $B = N \times C_G(N)$. For the proof of Theorem 1.2 we will distinguish the abelian socle case (1) from the non-abelian socle cases (2) and (3).

The next lemma shows that in the abelian socle case (1), G is of affine type as in part (1) of Theorem 1.2.

Lemma 3.1. *In case* (1)*, when N is abelian* (*so written additively*)*, it is divisible torsion-free. Furthermore, Ω can be identified with N, and $G = N \rtimes L$, where L is the stabiliser of the* 0 *of N. The group N has the structure of a* **Q***-vector space, with L acting as a group of linear transformations, and there are no definable proper non-trivial L-invariant subgroups of N.*

Proof. As N acts regularly on Ω, to identify N with Ω, fix $\alpha \in \Omega$, and for each $\beta \in \Omega$ identify β with the unique $n \in N$ such that $\alpha^n = \beta$. Now let L be the stabiliser in G of 0. It is standard that G is the semi-direct product $N \rtimes L$, and that the action of L on N by conjugation is isomorphic to the action of L on Ω, under the above identification. Also, any L-invariant subgroup of N is a block of imprimitivity for G. For any $k \in \mathbf{N}$, kN is a definable characteristic subgroup of N, and so by the above is $\{0\}$ or N. Likewise, for any $k \in \mathbf{N}$, $\{x \in N : kx = 0\}$ equals $\{0\}$ or all of N. We must show that $\{x \in N : kx = 0\} = \{0\}$. For then N is divisible torsion-free and L acts **Q**-linearly on N.

First suppose that G^o is centreless. Then, by [**32**, Proposition 2.3] G^o and also G have no infinite abelian subgroups of finite exponent.

If $Z = Z(G^o) \neq \{1\}$, then Z is normal in G and abelian, and so regular, and hence $Z = N$. As G acts faithfully on Ω, $G^o = Z$. Hence $|G : Z|$ is finite, so the orbits of G on N are finite. Either N is divisible and torsion free as before, or N has finite exponent and thus is a vector space over a finite field. In this case there will be non-trivial finite, so definable, G-invariant subspaces of N, yielding a contradiction.

In the rest of this section, we complete the O'Nan–Scott reduction for (G, Ω) in cases (2) and (3) when N is non-abelian. Since the argument is almost identical to that of [**11**] (itself very similar to that of [**10**] and [**12**]), we omit details, just indicating how the different cases arise.

Assume that case (2) or (3) holds. Then N has no non-trivial definable abelian normal subgroup: indeed, the G-conjugates of a minimal such group would generate an abelian normal subgroup M of G, and $N \cap C_G(C_G(M))$ would be a definable normal subgroup of G which contains M and is abelian, and so is properly contained in N, contradicting the minimality of N. Hence, if A is any abelian normal subgroup of N then $A = 1$ because $C_N(C_N(A))$ is a *definable* abelian normal subgroup of N containing A.

It follows from Theorem 2.5 that there are a positive integer m and definable real closed fields $R_1, \ldots, R_m$ such that $N = T_1 \times \ldots \times T_m$, where each T_j is a definable normal subgroup of N which is definably simple and definably isomorphic to a semialgebraic subgroup of $\mathrm{GL}(n, R_j)$. Furthermore, by [**20**, Theorem 1.1], T_j is bi-interpretable with $(R_j, +, .)$ or $(R_j(\sqrt{-1}), +, .)$. As N is a *minimal* definable normal subgroup of G, all the T_j are G-conjugate, so either each is bi-interpretable with a fixed real closed field R, or each is bi-interpretable with a fixed algebraically closed field $R(i)$. In particular, all the T_j are definably isomorphic to some definably simple group T, with $T \leqslant \mathrm{GL}(n, R)$ and bi-interpretable with R or $R(i)$. In particular, R is definable in $\mathscr{M}$, and all the T_j are semialgebraic over R (up to definable isomorphism). By the isomorphism of N and $C_G(N)$, in case (3), $C_G(N) = S_1 \times \ldots \times S_m$, where each S_j is $(\mathbf{G}, \Omega)$-definably isomorphic to T.

Fix $\alpha \in \Omega$. Let $B = T_1 \times \ldots \times T_k$ (so in case (3), $k = 2m$ and $S_j = T_{m+j}$ for each $j = 1, \ldots, m$). We may assume that $k > 1$, since otherwise case (2) of Theorem 1.2 holds. For $i = 1, \ldots, k$ let π_i be the projection of B onto T_i, and $Q_i := \pi_i(B_\alpha)$. As in [**11**, Lemma 2.4], B_α is a maximal definable proper (possibly trivial) G_α-invariant subgroup of B.

Case A: $Q_i = T_i$ for some $i \in \{1, \ldots, k\}$.

This is handled as in case (1) of the proof of [**11**, Theorem 1.1]. First, one shows that $Q_i = T_i$ for *all i*. Then, since the T_i are definably simple, there is a partition $\{1, \ldots, k\} = I_1 \cup \ldots \cup I_\ell$ and for each $i = 1, \ldots, \ell$ a definable diagonal subgroup D_i of $\prod(T_j : j \in I_i)$, such that $B_\alpha = D_1 \times \ldots \times D_\ell$. Here, if $N \neq B$ then D_i is a diagonal subgroup of $T_i \times T_{m+i}$. If now $\ell = 1$, then G has the *simple diagonal action* on Ω, as in case (3) of Theorem 1.2. On the other hand, if $\ell > 1$, then the action is a wreath product of simple diagonal actions (and satisfies case (4)(ii) of Theorem 1.2).

Case B: For each $i \in \{1, \ldots, k\}$, Q_i is a proper subgroup of T_i.

In this case, $B_\alpha = Q_1 \times \ldots \times Q_k$ where all the Q_i are definably isomorphic. We now follow the arguments in cases 2(a) and 2(b) of [**10**] (see also [**11**, case 2 of the proof of Theorem 1.1]). Note here that the o-minimal version of the 'Schreier conjecture' holds: the outer automorphism group of a definably simple group is soluble (this uses [**30**, Theorem 30; **20**, Theorem 3.23], and see Section 5 below). It follows, as in [**10**], that case 2(a) of [**10**] cannot hold, for otherwise there would be a homomorphism from a subgroup of S_k onto the group of inner automorphisms of T_1, which is clearly impossible. This eliminates the 'twisted wreath product' case of finite primitive permutation groups. In case 2(b), as in [**10**], one shows that the action of G on Ω is of type 4(i).

This completes our sketch of the proof of Theorem 1.2.

4. *The abelian socle case*

We now turn towards the proof of Theorem 1.1. Throughout the section, we work in an ambient o-minimal structure $\mathscr{M}$, and 'definable' means 'definable in $\mathscr{M}$'. The main step is the abelian socle case, so our goal here is to prove Proposition 1.4. We first re-state Proposition 1.4 in an equivalent form, not using the language of permutation groups (and with A in place of N).

PROPOSITION 4.1. *Let G be a group definable in $\mathcal{M}$, and suppose that G has a non-trivial definable torsion-free divisible abelian normal subgroup A, and an infinite definable subgroup L, such that $G = A \rtimes L$ and $C_G(A) = A$. Suppose too that, in the action of L on A by conjugation in G, there are no L-invariant proper non-trivial definable subgroups of A. Then there is a definable real closed field R such that A has definably the structure of a vector space over R of finite dimension m say, and L is definably isomorphic to a subgroup L^* of $\mathrm{GL}(m, R)$, acting R-linearly on A by conjugation. Furthermore, if G is connected then L^* is R-semialgebraic.*

We need the following lemma.

LEMMA 4.2. *Let R be a definable real closed field, V be a definable finite-dimensional R-vector space, and H be a definable irreducible subgroup of $\mathrm{GL}(V)$. Let M be a definable connected nilpotent normal subgroup of H. Then $M \leqslant Z(H)$.*

Proof. Let W be a non-trivial simple RM-submodule of V. By Clifford's theorem and connectedness of H, V is a homogeneous RM-module, so $V = W_1 \oplus \ldots \oplus W_s$ for some $s \in \mathbf{N}$, where the W_i are isomorphic as RM-modules to W. In particular, the W_i are simple non-trivial and faithful RM-modules. By [**31**, Theorem 27], if k is any field, then any irreducible nilpotent subgroup of $\mathrm{GL}(n, k)$ is a finite extension of its centre. As M is connected, it follows that M is abelian.

By irreducibility, the subring of $\mathrm{End}(W)$ generated by RM is an integral domain K, say. Also, if $\dim_R(W) = t$, then for $u, v \in W \setminus \{0\}$ there are $r_1, \ldots, r_t \in R$ and $m_1, \ldots, m_t \in M$ such that $(r_1 m_1 + \ldots + r_t m_t) u = v$ (regarding W as an RM-module). It follows that K is interpretable in $\mathcal{M}$. Since all the data of the lemma lives in an o-minimal expansion of R, we may suppose that K is *definable* in $\mathcal{M}$. Hence, since M is abelian, by Theorem 2.1, K equals R or $R(i)$. As $M \lhd H$, H induces a group of automorphisms of K, but since H is connected and complex conjugation is the only automorphism of $R(i)$, it follows that H centralises K. In particular, $M \leqslant Z(H)$.

Proof of Proposition 4.1. We shall write A additively. Since A has no proper non-trivial definable G-invariant subgroups, $G^o \cap A = A$. Thus $G^o = A \rtimes L^o$. Hence G is connected if and only if L is. In the following argument, we shall not assume connectedness of G until the final paragraph.

The group G^o is connected and also centreless, for since $C_G(A) = A$, we have $Z(G^o) \leqslant A$, and since L^o acts faithfully on A by conjugation and $Z(G^o)$ is L-invariant, we get $Z(G^o) = 1$. Hence, by Theorem 2.4, there are a positive integer k and positive integers $n_1, \ldots, n_k$, definable real closed fields R_i and definable groups $H_i \leqslant \mathrm{GL}(n_i, R_i)$ (for $i = 1, \ldots, k$) such that G^o is definably isomorphic to $H_1 \times \ldots \times H_k$. The structure induced on each R_i is an o-minimal expansion of an ordered field. Just as in the usual proof that o-minimal expansions of ordered fields have elimination of imaginaries, any $\mathcal{M}$-interpretable set living in $(R_1 \times \ldots \times R_k)^p$ can be identified with a set *definable* in $\mathcal{M}$. In particular, definable sections of G^o can be regarded as groups *definable* in $\mathcal{M}$, so facts from Section 2 about groups definable in o-minimal structures can be applied to them.

By the descending chain condition, there is a minimal non-trivial L^o-invariant definable subgroup U of A, possibly with $U = A$. Let $H := L^o / C_{L^o}(U)$, which embeds in $\mathrm{Aut}(U)$. By the last paragraph, H can be identified with a group *definable* in $\mathcal{M}$. Clearly, H acts $\mathbf{Q}$-linearly on U.

Claim 1. H is infinite.

Proof of Claim 1. Let W be a maximal direct sum of L-translates of U. If X is a group definable in an o-minimal structure, then $\dim(X) > \dim(Y)$ for any definable subgroup Y of X of infinite index (see [**23**, Lemma 2.3]). It follows (as any finite direct summand of L-conjugates of U is definable) that W is a finite direct sum of L-translates of U, so is definable. As W is L-invariant, it follows that $W = A$. Thus we may write $A = U_1 \oplus \ldots \oplus U_t$ where $U_1 = U$ and there are $g_i \in L$ satisfying $U^{g_i} = U_i$ for each $i = 1, \ldots, t$. Let $K = C_L(U)$. If H is finite, then $|L:K|$ is finite, so $|L:\bigcap_{i=1}^{t} K^{g_i}|$ is finite. Since $\bigcap_{i=1}^{t} K^{g_i} = 1$, it follows that L is finite, a contradiction.

Put $F := U \rtimes H$. Clearly F is centreless and connected. Furthermore, F cannot be written non-trivially as a direct product. It follows by Theorem 2.4 that there is a definable real closed field R such that F is definably isomorphic to a subgroup of $\mathrm{GL}(n, R)$ for some n.

Our next aim is to show that F is definably isomorphic to an R-semialgebraic group. For convenience we assume now, until the end of Claim 6, that F is in fact a subgroup of $\mathrm{GL}(n, R)$ (so 'R-semialgebraic' should be read, without this assumption, as 'definably isomorphic to an R-semialgebraic'). By Theorem 2.6, there is a definable normal subgroup B of F such that B is R-semialgebraic and F/B is abelian.

Claim 2. $U \leqslant B$.

Proof of Claim 2. If the claim is false, then by definable minimality of U, $U \cap B = 1$, so $[U, B] = 1$, contrary to the assumption that H acts faithfully on U.

Claim 3. U and $B \cap H$ are R-semialgebraic.

Proof of Claim 3. To see that U is semialgebraic it suffices to observe that $U = C_B(U)$, and that centralisers in B are semialgebraic. Also, $B \cap H$ is just the stabiliser of 0 in the semialgebraic action of B on U and thus clearly semialgebraic.

Claim 4. U is R-definably isomorphic to $(R, +)^{\ell}$ for some ℓ, and H acts R-linearly on U.

Proof of Claim 4. By Claim 3, U is a subgroup of $\mathrm{GL}(n, R)$ definable in the pure field language of the real closed field R. Thus there is some abelian linear algebraic group $\bar{U} \leqslant \mathrm{GL}(n, R(i))$ defined over R, such that U is a Zariski-open subgroup of $\bar{U}(R)$ (cf. [**24**, Remark 2.6]). Now the Jordan decomposition (see for example [**1**, 4.7]) yields a direct product $\bar{U} = \Delta \times U_u$ where Δ consists of semi-simple elements, and so is diagonalisable in $\mathrm{GL}(n, R(i))$, and U_u is unipotent, and both Δ and U_u are defined over R. Hence $U \leqslant \bar{U}(R) = \Delta(R) \times U_u(R)$ (see [**7**, Section 34.2, 15.3]).

First observe that if $U_u(R) \cap U$ is non-trivial, then it is infinite, and since H acts on U by conjugation inside $\mathrm{GL}(n, R)$, $U_u(R) \cap U$ is H-invariant, so $U \leqslant U_u(R)$. Similarly, if $\Delta(R) \cap U$ is non-trivial, then $U \leqslant \Delta(R)$.

Now suppose that $U \leqslant \Delta(R)$. Then as Δ is the Zariski closure of U, and conjugation by elements of H is continuous (working in $\mathrm{GL}(n, R(i))$), elements of H normalise Δ. It follows that $N_{\mathrm{GL}(n, R(i))}(\Delta)$ induces an infinite group of automorphisms of Δ, contrary to rigidity [**1**, III.8, Corollary 2, p. 117].

A similar argument shows that U cannot have trivial intersection with both $U_u(R)$ and $\Delta(R)$, for otherwise the projection of U to $\Delta(R)$ would be dense in $\Delta(R)$ and H-invariant, again leading to a contradiction. Thus $U \leqslant U_u(R)$.

It follows that U is conjugate in $\mathrm{GL}(n, R(i))$ to a group of upper uni-triangular matrices, and we may suppose that U consists of upper triangular matrices. As in [**7**, Section 15, Exercise 8] the map $x \longmapsto \log(x)$ gives a bijection (given by a polynomial, so definable) between the group of unipotent upper triangular matrices and the additive group Y of nilpotent upper triangular matrices, which has the structure of an $R(i)$-vector space. As U is abelian, $\log(U)$ is an additive subgroup of Y. The set $\{a \in R : a \log(U) \subseteq \log(U)\}$ is a definable subring of R so must be R by Theorem 2.1. Thus, $\log(U)$ is an R-vector space. Also, the isomorphism $\exp : \log(U) \longrightarrow U$ is definable. Thus, using exp, we can pull back the R-vector space structure to U, that is, if $\exp(\tilde{u}) = u$, then set $r \cdot u = \exp(r \cdot \tilde{u})$. It follows that U has definably in R the structure of an R-vector space of finite dimension ℓ, say. By Lemma 2.2, $H \leqslant \mathrm{GL}(\ell, R)$ in its action on U.

Since U is finite-dimensional over R, every R-subspace of U is definable. Hence H is irreducible on U.

Claim 5. $H/Z(H)$ is a direct product $H_1 \times \ldots \times H_t$ of definable non-abelian definably simple groups.

Proof of Claim 5. Let E be any definable nilpotent normal subgroup of $H/Z(H)$, with preimage $\bar{E}$ in H. By applying Lemma 4.2 to $(\bar{E})^o$ we see that $\bar{E}$ is a finite extension of $Z(H)$, and by connectedness of H, $E \leqslant Z(H/Z(H))$.

In particular, we may put $E = Z(H/Z(H))$. By the last paragraph, there is no non-trivial definable abelian normal subgroup of $H/\bar{E}$, so by Theorem 2.5 and our elimination of imaginaries for H, $H/\bar{E}$ is a direct product of definable non-abelian definably simple groups. Thus it suffices to show that E is trivial.

If E is non-trivial, then by connectedness H induces an infinite group of automorphisms of $\bar{E}$. Furthermore, since H centralises $Z(H)$ and $E = \bar{E}/Z(H)$, the group of automorphisms of $\bar{E}$ induced by H is abelian (for it embeds into a cartesian power, indexed by E, of copies of $Z(H)$). However, $\bar{E}$ induces a finite group of inner automorphisms (as $\bar{E}/Z(\bar{E})$ is finite), and $H/\bar{E}$ has no abelian composition factors. This is a contradiction.

Claim 6. $F = U \rtimes H$ is an R-semialgebraic group (assuming that $F \leqslant \mathrm{GL}(n, R)$).

Proof of Claim 6. Put $Y := B \cap H$. Since $H/Z(H)$ is a product of non-abelian definably simple groups and H/Y is abelian, we have $H = Y * Z(H)$ (a central product). Since $\mathrm{GL}(\ell, R)$, in its action on U, is R-semialgebraic, it suffices to show that H is an R-definable subset of $\mathrm{GL}(\ell, R)$. For this, since Y is R-definable, we must show that $Z(H)$ is R-definable. We may assume that $Z(H)$ is infinite, since otherwise this is obvious. By Schur's lemma, we know that $Z(H)$ acts as scalars contained in the centre of $D = C_{\mathrm{End}(U)}(H)$. The quotient field of the ring generated by $RZ(H)$ is an infinite definable commutative subfield of D. Thus $Z(H)$ is a definable subgroup of the multiplicative group of R or $R(i)$. In the first case it follows that $Z(H) = R^*$ or $Z(H) = R_{>0}$, so $Z(H)$ is semialgebraic.

In the second case, if $Z(H) = R(i)^*$, this is again semialgebraic. Suppose now that $Z(H) < R(i)^*$ and $\dim Z(H) = 1$. Then we have the usual definable homomorphism

$$\varphi: Z(H) \longrightarrow \mathrm{SO}(2, R) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a, b \in R, a^2 + b^2 = 1 \right\}$$

with $\varphi(x) = x/|x|$ and the other definable homomorphism $\varphi: Z(H) \longrightarrow R_{>0}$ with $\psi(x) = |x|$ (here, for $x = a + ib$ we set $|x| := c \geqslant 0$ where $c^2 = a^2 + b^2$). By dimension, if φ or ψ is surjective then its kernel is finite. Since $\mathrm{SO}(2, R)$ is a connected group of dimension 1, we must have either $\varphi(Z(H)^o) = \{1\}$, in which case $Z(H)^o = R_{>0}$, or $\varphi(Z(H)^o) = \mathrm{SO}(2, R)$. In the second case, we similarly have either $\psi(Z(H)^o) = \{1\}$, in which case $Z(H)^o = \mathrm{SO}(2, R)$, or $\psi(Z(H)^o) = R_{>0}$. Clearly, if $Z(H)^o$ is semialgebraic, then so is $Z(H)$. Hence, we may suppose for a contradiction that both φ and ψ are surjective. Since $R_{>0}$ is torsion-free, it follows that the torsion subgroup of $Z(H)$ is finite. In $\mathrm{SO}(2, R)$ the torsion elements are dense, so if $\varphi(Z(H)) = \mathrm{SO}(2, R)$ and $\psi(Z(H)) = R_{>0}$, then $\ker \varphi$ is an infinite definable subgroup of $Z(H)$, which is impossible. Thus, $Z(H)$ is either the isotropic torus $R_{>0}$ or the anisotropic torus $\mathrm{SO}(2, R)$ and is therefore semialgebraic.

Dropping our assumption that F is a subgroup of $\mathrm{GL}(n, R)$, we have now shown that F is definably isomorphic to a semialgebraic subgroup of $\mathrm{GL}(m, R)$. If G is connected, then $F = G$, and the proof is finished. We now drop the connectedness assumption, and apply the proof of Clifford's theorem. As in the proof of Claim 1 above, A is a direct sum of finitely many L-translates of U; that is, A can be definably equipped with a vector space structure of finite dimension m over R, with L^o acting R-linearly. Finally, by Lemma 2.2, $L \leqslant \mathrm{GL}(m, R)$, as required.

REMARK 4.3. (1) By Clifford's theorem, we can write $A = W_1 \oplus \ldots \oplus W_r$, where each W_i is a homogeneous RL^o-module of dimension s say, and $L \leqslant \mathrm{GL}(r, R) \operatorname{wr} S_r$, with L inducing a transitive subgroup of S_r on $\{W_1, \ldots, W_r\}$.

(2) If R is uncountable and H is finite, then by cardinality considerations there are G-invariant subgroups of V. This gives many examples of definably primitive groups which are not primitive. The simplest example of the last phenomenon is the additive group of $\mathbf{Q}$ acting regularly on itself. This is up to elementary equivalence the only example of a definably primitive *regular* action.

Claims 5 and 6 of the above proof very easily yield the following result.

PROPOSITION 4.4. *If R is a definable real closed field in $\mathcal{M}$, and H is a connected definable subgroup of* $\mathrm{GL}(n, R)$ *acting irreducibly on $V = R^n$, then H is R-semialgebraic.*

5. *The non-abelian socle case*

In the previous section we proved Theorem 1.1 for the abelian socle case. We now complete the proof of Theorem 1.1, by handling the case of non-abelian socle, that is, groups of type 2, 3 or 4 in Theorem 1.2. The proof uses the description of G provided by Theorem 1.2.

PROPOSITION 5.1. *Assume that* (G, Ω) *is of type* 2, 3 *or* 4 *in Theorem* 1.2. *Then the conclusion of Theorem* 1.1 *holds.*

Proof. Using Theorem 1.2 (and its notation) and Remark 1.3 we see that G^o is a direct product $T_1 \times \ldots \times T_k$, where the groups T_i are all definably isomorphic to a definably simple group T. The group T can be taken to be a semialgebraic group over a definable real closed field R (by Theorem 2.5), and is bi-interpretable with R or $R(i)$ (by [**20**, Theorem 1.1]). It follows that G^o is definably isomorphic to an R-semialgebraic group.

We claim that G is definably isomorphic to a semialgebraic group over R. Since $|G: T_1 \times \ldots \times T_k|$ is finite, to prove the claim it suffices to show that any definable automorphism of T is semialgebraic over R, for then there is an R-semialgebraic group structure on an R-semialgebraic set $T^k \times S$ (where $|S| = |G:G^o|$), and a definable isomorphism between G and this semialgebraic group. If T is bi-interpretable with $R(i)$, then every automorphism is a product of diagonal automorphisms, field automorphisms, and graph automorphisms (see [**30**, Theorem 30]). Since the only non-trivial definable field automorphism is conjugation, the result follows in this case. On the other hand, if T is bi-interpretable with a real closed field, then by [**20**, Proposition 3.23], every automorphism of T is a product of a semialgebraic automorphism and a field automorphism. Again, as the field automorphism must be trivial, the claim follows.

We must now show that the action of G on Ω is semialgebraic. Let $H := G_\alpha$. We identify Ω with the coset space G/H. Thus, since $(R, +, .)$ has elimination of imaginaries, it suffices to show that H is definable in R. We may suppose that H is infinite, since otherwise it is clearly definable.

We use Peterzil's argument as in the proof of [**32**, Lemma 2.7]. The tangent space $\mathcal{T}_e(G)$ carries the structure of a Lie algebra $L(G)$. Let $L(H)$ be the Lie algebra of H. Let Ad denote the adjoint action of G on $L(G)$. Now by [**19**, 2.19 and 2.30(2)], the normaliser of H^o in G is $M := \{g \in G : \mathrm{Ad}(g)\, L(H) \subseteq L(H)\}$. Thus, M is semialgebraic so definable, and contains H. Since H^o is intransitive (and G is not regular), $M \neq G$. Hence, as H is definably maximal in G, $M = H$ and H is definable in R.

This concludes the proof of Theorem 1.1.

6. *Groups acting on a* 1-*dimensional set*

In the next three sections, we prove Theorem 1.5. Section 6 contains some general reductions. Unless stated otherwise, the assumptions of Theorem 1.5 are assumed to hold throughout Sections 6–8.

By Theorem 2.3, we may suppose that G and Ω carry a manifold structure, with G having a $C^{(p)}$-action on Ω. (Here $p \geqslant 0$, and we may suppose that $p > 0$ if the ambient structure expands an ordered field.) This manifold structure on Ω is definably connected, for it has finitely many definably connected components, permuted transitively by G, and if there is more than one component then the subgroup fixing setwise each component is a definable proper subgroup of finite index, contrary to connectedness.

We now apply the classification by Raženj [**26**, Proposition 2] of the definably connected 1-dimensional Hausdorff manifolds. One of the following holds, where $x \in \Omega$ is fixed.

(1) $\Omega \backslash \{x\}$ has two definably connected components, and the definably connected open subsets of Ω (in its manifold topology) are given by the interval topology of a definable total ordering $<$ on Ω.

(2) $\Omega\backslash\{x\}$ is definably connected, and there is a definable linear ordering $<$ on $\Omega\backslash\{x\}$ whose open intervals are the definably connected open sets in the induced topology on $\Omega\backslash\{x\}$.

In both cases, the total orderings mentioned are obtained by piecing together finitely many subsets in definable bijection with intervals of $\mathcal{M}$. In particular, the total orderings mentioned are o-minimal, in the sense that any subset of them definable (in the ambient structure) is a finite union of intervals. In particular, the monotonicity theorem of [**25**] can be applied to definable functions on the linear orderings. Hence, in case (1), as elements of G are homeomorphisms they act monotonically, and as G is connected they are order-preserving, that is, $G \leqslant \mathrm{Aut}(\Omega, <)$. In case (2), for fixed $x \in \Omega$ there is a definable ternary relation K on Ω, given as follows: $K(u, v, w)$ holds if and only if one of the following holds:

(a) $u = x \wedge v < w$;
(b) $v = x \wedge w < u$;
(c) $w = x \wedge u < v$;
(d) u, v, w, x are distinct and $(u < v < w) \vee (w < u < v) \vee (v < w < u)$.

Again, because of the monotonicity theorem, $G < \mathrm{Aut}(\Omega, K)$. The relation K is a *cyclic ordering*, that is, it is invariant under cyclic permutations of the arguments and induces a linear ordering $\prec$ on $\Omega\backslash\{x\}$, where we put $u \prec v$ whenever $K(x, u, v)$ holds.

To prove Theorem 1.5, it now suffices to prove the following two propositions. The first does not require connectedness.

PROPOSITION 6.1. *Suppose that (G, Ω) is a definable transitive permutation group in an o-minimal structure preserving a definable linear order $<$ on Ω, that any definable subset of Ω is a finite union of intervals, and that* $\dim(\Omega) = 1$. *Then one of* (1) (*with* $G = \bigoplus_{\delta\in\Delta} \mathbf{Q}$), *or* (2) *of Theorem* 1.5 *holds.*

PROPOSITION 6.2. *Suppose that (G, Ω) is as in Theorem* 1.5, *and there is no definable G-invariant linear order on Ω. Then there is a definable G-invariant cyclic ordering K on Ω, and one of* (1), (3) *of Theorem* 1.5 *holds.*

The following lemma is used in the proofs of both propositions.

LEMMA 6.3. *Let (G, Ω) be a definable transitive infinite permutation group in $\mathcal{M}$. For $x, y \in \Omega$, write $x \sim y$ if G_x fixes y. Then $\sim$ is a G-invariant equivalence relation on Ω.*

Proof. It suffices to show that $\sim$ is symmetric. If G_x fixes y and G_y does not fix x, then $G_x < G_y$, and as these groups are conjugate (by transitivity), this contradicts the descending chain condition on definable subgroups.

7. *The linear case*

In this section, we give a proof of Proposition 6.1. We assume throughout the section that the assumptions of Proposition 6.1 hold.

LEMMA 7.1. *The group G is definably primitive on Ω.*

Proof. Suppose that E is a definable G-invariant equivalence relation on Ω, which is not equality. By transitivity, all the E-classes have the same size, and by dimension considerations [**23**, Proposition 1.8], either there are finitely many E-classes, or all the E-classes have the same finite size. The latter is impossible, since the group induced on each E-class would preserve the total ordering, and so be trivial, contrary to transitivity. On the other hand, if there is a finite number greater than one of the E-classes, then each of these classes is a finite union of (maximal) intervals, and the endpoints of these intervals are fixed, which is also impossible. Thus, there is a unique E-class.

LEMMA 7.2. *If G is not regular on Ω, then it is 2-homogeneous.*

Proof. Since G is not regular, we may assume that G_x acts non-trivially on $\Gamma \coloneqq \{y \in \Omega : x < y\}$. Let Δ be an orbit of G_x on Γ. Then Δ is a finite union of maximal intervals, and any endpoint of these intervals is fixed by G_x. However, by Lemmas 7.1 and 6.3, G_x has no fixed points other than x. It follows that $\Delta = \Gamma$, so (G, Ω) is 2-homogeneous.

LEMMA 7.3. *For some $n \geqslant 1$, (G, Ω) is sharply n-homogeneous.*

Proof. First observe that G cannot be k-homogeneous for all $k \in \mathbf{N}$, since otherwise there would be an infinite descending chain $G, G_{x_1}, G_{x_1 x_2}, \ldots$ of definable subgroups. Thus, to prove the lemma it suffices to show that if for some k the group G is k-homogeneous but not sharply k-homogeneous on Ω, then G is $(k+1)$-homogeneous. We prove this by induction on k, the induction being started by Lemma 7.2.

We now suppose that G is k-homogeneous but not sharply k-homogeneous for some $k \geqslant 2$, fix $x \in \Omega$, and put $\Gamma = \{y \in \Omega : x < y\}$. Then G_x is $(k-1)$-homogeneous on Γ.

We claim that (G_x, Γ) is not *sharply* $(k-1)$-homogeneous. Suppose otherwise, and pick $x_1 < \ldots < x_{k-1}$ in Γ. Then $G_{x x_1 \ldots x_{k-1}}$ fixes Γ pointwise, but moves some $y < x$. It follows by k-homogeneity that $G_{y x_1 \ldots x_{k-1}}$ fixes $\{z : y < z\}$, so $G_{y x_1 \ldots x_{k-1}} < G_{x x_1 \ldots x_{k-1}}$. Since these groups are conjugate, this contradicts the descending chain condition on definable subgroups.

It follows by induction that (G_x, Γ) is k-homogeneous, and hence (G, Ω) is $(k+1)$-homogeneous, as required.

LEMMA 7.4. *If G is regular on Ω, then G is a divisible ordered abelian group.*

Proof. In this case, we may identify $(G, <)$ with $(\Omega, <)$. Since any definable subset of Ω is a finite union of intervals, the result follows from the fact, proved in [**25**], that any o-minimal ordered group is divisible abelian.

COROLLARY 7.5. *The group G is connected.*

Proof. If G is regular then as in Lemma 7.4, G is an o-minimal ordered group, so has no proper non-trivial definable subgroup. On the other hand, if G is not regular then by Lemma 7.3 it is sharply k-homogeneous for some $k > 1$. If N is a definable normal subgroup of finite index, then N is sharply ℓ-homogeneous for some $\ell \leqslant k$, and since N has finite index in G, we must have $\ell = k$, so $N = G$.

Proof of Proposition 6.1. By Lemma 7.4, we may suppose that G is not regular. Hence (G, Ω) is definably primitive, connected, and not regular, so by Theorem 1.1 there is a definable real closed field R and an R-semialgebraic permutation group which we definably identify with (G, Ω).

The facts that $\dim(\Omega) = 1$ and that G is 2-homogeneous and preserves a linear order on Ω are first-order expressible (with parameters) in R, by some formula $\phi(\bar{x}, \bar{a})$, say. It follows by transfer that the formula $\exists \bar{y}(\phi(\bar{x}, \bar{y})$ defines a 2-homogeneous permutation group on a 1-dimensional set and preserves a linear order) is part of $\mathrm{Th}(R, +, .)$, and so holds in $(\mathbf{R}, +, .)$. Let $(G^*(\bar{b}), \Omega^*(\bar{b}))$ be any instance of this sentence in $\mathbf{R}$ (so the witness for $\bar{y}$ is $\bar{b}$). Then, over $\mathbf{R}$, by Theorem 2.3 there is a definable *analytic* manifold structure on $\Omega^*(\bar{b})$, on which $G^*(\bar{b})$ acts analytically. Furthermore, by Corollary 7.5, $G^*(\bar{b})$ is connected in the o-minimal sense, and hence also topologically connected. It follows by Brouwer's theorem ([**27**, 96.30], see also [**2**, II, Theorem 2.1, p. 218]) that $G^*(\bar{b})$ is isomorphic to the affine group $\mathbf{R} \rtimes \mathbf{R}_{>0}$ in its natural action on $\mathbf{R}$. The group $G^*(\bar{b})$ is therefore bi-interpretable with a copy of $\mathbf{R}$, which is definable in $\mathbf{R}$ and hence $\mathbf{R}$-definably isomorphic to $\mathbf{R}$. It follows that the permutation group $(G^*(\bar{b}), \Omega^*(\bar{b}))$ is uniformly (in $\bar{b}$) definably isomorphic to $(\mathbf{R} \rtimes \mathbf{R}_{>0}, \mathbf{R})$, for otherwise in some elementary extension $(R', +, .)$ of $(\mathbf{R}, +, .)$ there would be a definable real closed field not definably isomorphic to $(R', +, .)$, contrary to [**16**]. It follows by transfer that (G, Ω) is R-definably isomorphic to $R \rtimes R_{>0}$ in its natural action on R, as required.

8. *The cyclic case*

Now we consider definable transitive permutation groups (G, Ω) where Ω is a set of dimension 1 equipped with a definable cyclic ordering K, and $G \leqslant \mathrm{Aut}(\Omega, K)$ and is definably connected. In addition, Ω is definably connected, and for any $\alpha \in \Omega$, in the G_α-invariant total ordering induced on $\Omega \setminus \{\alpha\}$, any definable set is a finite union of intervals. There is a topology on Ω with a uniformly definable basis, with respect to which G acts as a group of homeomorphisms. The basic open sets are the sets $\{x \in \Omega : K(\alpha, x, \beta)\}$ where $\alpha, \beta \in \Omega$.

By connectedness of G, any proper definable G-invariant equivalence relation on Ω has infinitely many classes, so by dimension considerations, all the classes will be finite. In fact, we have a precise description for the unique greatest such equivalence relation. By Lemma 6.3, there is a G-invariant equivalence relation E, where $E\alpha\beta$ holds if and only if G_α fixes β. Furthermore, since G_α preserves a total order on $\Omega \setminus \{\alpha\}$, all its non-trivial orbits are infinite, so E is the largest proper G-invariant equivalence relation. Let n be the size of the E-classes. We denote by x_E the E-class of x. Put $\Delta := \{x_E : x \in \Omega\}$.

LEMMA 8.1. *Each infinite orbit of G_α contains exactly one member of each equivalence class other than α_E.*

Proof. Suppose first that there are distinct x, y in an infinite orbit of G_α, with Exy. Then there is $g \in G_\alpha$ with $x^g = y$. Now g fixes the finite set x_E setwise. Hence, as g preserves a total order on $\Omega \setminus \{\alpha\}$, g fixes x_E pointwise, a contradiction.

By considering endpoints, the number of infinite orbits of G_α is equal to n, the number of fixed points. Since each E-class has size n, the lemma follows.

Let Δ be the set of E-classes. Consider the interval $[\alpha, \alpha')$ in Ω, where α' is the least fixed point of G_α in $\Omega \backslash \{\alpha\}$. Then there is exactly one member of each equivalence class in $[\alpha, \alpha')$. Define the ternary relation K' on Δ as follows: $K'(x_E, y_E, z_E)$ holds if and only if $K(x_E \cap [\alpha, \alpha'), y_E \cap [\alpha, \alpha'), z_E \cap [\alpha, \alpha'))$ holds. It is routine to verify that K' defines a G-invariant cyclic ordering on Δ. Furthermore, G acts definably primitively on Δ, and the kernel of this action (which must act faithfully on each E-class) is cyclic of order n.

The group $\bar{G}$ induced on Δ is definably connected, since G is. From the above description, it is immediate that $(G_\alpha, \Delta \backslash \{\alpha_E\})$ is transitive, so $(\bar{G}, \Delta)$ is 2-transitive. By connectedness of G, the kernel of the action of G on Δ is central, and since any doubly transitive permutation group on a set of size bigger than 2 is centreless (or by Theorem 1.2), this kernel equals $Z(G)$.

Proof of Proposition 6.2. We may suppose that (G, Ω) is not regular, since otherwise the result follows from the classification of 1-dimensional groups in o-minimal structures, given in [**26**]. Since $\bar{G} = G/Z(G)$ and $Z(G)$ is finite, $\bar{G}$ is in interpretable bijection with a *definable* group $\bar{H}$, acting 2-transitively and definably on a cyclically ordered set (Γ, K). (This uses the fact that in an o-minimal structure we can eliminate imaginaries given by an equivalence relation with finite classes, since each class (of tuples) is coded by a tuple listing the tuples in the class in increasing lexicographic order.)

The group (H, Γ) is a definable 2-transitive permutation group, with $\dim(\Gamma) = 1$. We must show that there is a definable real closed field R such that (H, Γ) is definably isomorphic to $\mathrm{PSL}(2, R)$ acting on the projective line. There are various ways to obtain this, but the easiest seems to be by transfer from $\mathbf{R}$, as in the proof of Proposition 6.1. By Theorem 1.1, we may suppose that (H, Γ) is semialgebraic in a definable real closed field R. It follows that there is a 2-transitive $\mathbf{R}$-semialgebraic permutation group (H^*, Γ^*) with $\dim(\Gamma^*) = 1$ (in the structure $(\mathbf{R}, +, .)$). If $x \in \Gamma^*$ then H^*_x is connected by Corollary 7.5. Hence, if N is a definable normal subgroup of H^* of finite index then $H^*_x \leqslant N$, so as $H^* = H^*_x N$, we have $N = H^*$. Thus, H^* is connected. It follows by Brouwer's theorem [**27**, 96.30] that (H^*, Γ^*) is isomorphic to $\mathrm{PSL}(2, \mathbf{R})$ acting on $\mathrm{PG}(1, \mathbf{R})$, and as in the proof of Proposition 6.1, this isomorphism is $\mathbf{R}$-definable, uniformly in the parameters (note that $\mathrm{PSL}(2, \mathbf{R})$ is bi-interpretable with $(\mathbf{R}, +, .)$). Hence, (H, Γ) is definably isomorphic to $(\mathrm{PSL}(2, R), \mathrm{PG}(1, R))$, as required.

Remark 8.2. For every n there is a semialgebraic example of a finite cover G_n of $\mathrm{PSL}(2, R)$ with centre cyclic of order n, such that G_n acts imprimitively on a cyclic order as above.

9. *Corollaries and further observations*

By Theorem 1.5, any definable transitive permutation group in an o-minimal structure, acting on a set of dimension 1, has dimension at most 3. We first show that there is no such corresponding bound in the 2-dimensional case. This is probably already well known (see [**27**, Remark 96.32]). The example below is based on one of Gropp [**3**] in the finite Morley rank case.

EXAMPLE 9.1. Let R be a real closed field. Let V be the direct sum of n copies of the additive group of R, and let G be a semi-direct product $V \rtimes R^*$, where the action of $t \in R^*$ on V is given by $(a_1, \ldots, a_n)^t = (ta_1, t^2 a_2, \ldots, t^n a_n)$. Let $B = \{(a_1, \ldots, a_n) \in V : a_1 + \ldots + a_n = 0\}$. Then B is a subgroup of G of dimension $n-1$, and the dimension of the coset space $(G:B)$ is 2. The group G acts transitively on this coset space, so it suffices to show that this action is faithful, that is, $\bigcap(B^t : t \in R^*) = \{0\}$. This is immediate, for given $(a_1, \ldots, a_n) \in B \setminus \{0\}$, the equation $\sum_{i=1}^n a_i x^i = 0$ has only finitely many solutions, so there is $t \in R^*$ with $\sum_{i=1}^n a_i t^i \neq 0$; then $(a_1, \ldots, a_n)^t \notin B$, so $(a_1, \ldots, a_n) \notin \bigcap(B^t : t \in R^*)$. However, this action is not definably primitive, since $B < V < G$.

We show now that the phenomenon described in Example 9.1 cannot arise in the definably primitive case, that is, the dimension of the group can be bounded in terms of that of the set. In contrast, for definably primitive groups of finite Morley rank, the analogue is not known.

THEOREM 9.2. *Let (G, Ω) be a definably primitive permutation group definable in an o-minimal structure. Then $\dim(G) \leqslant \dim(\Omega)^2 + 2\dim(\Omega)$ and this bound is sharp.*

Proof. We apply Theorem 1.2. Suppose first that (G, Ω) is of affine type. We may suppose the point stabiliser is infinite (as otherwise $\dim(G) = \dim(\Omega)$). Thus, Ω carries the structure of an n-dimensional vector space over R for some $n \in \mathbf{N}$, and then $\dim(\Omega) = n$ and as $G \leqslant V \rtimes \mathrm{GL}(n, R)$, we have $\dim(G) \leqslant n^2 + n$.

A short calculation shows that the bound holds for actions of simple diagonal types, that is, of type 3, for in the notation of Theorem 1.2(3), we have $\dim(\Omega) = (k-1)\dim(T)$ and $\dim(G) = k\dim(T)$ (see Remark 1.3). Furthermore, if the bound holds when (G, Ω) has type 2 (the definably simple socle case), then it also holds for type 4, for suppose (H, Γ) is a definably primitive permutation group with $\mathrm{Soc}_d(H)$ definably simple, so (by assumption) $\dim(H) \leqslant \dim(\Gamma)^2 + 2\dim(\Gamma)$. It follows that if G is of type 4, with $G \leqslant H \operatorname{wr} S_k$ acting on $\Omega = \Gamma^k$, then

$$\begin{aligned}\dim(G) = k\dim(H) &\leqslant k(\dim(\Gamma)^2 + 2\dim(\Gamma))\\ &\leqslant (k\dim(\Gamma))^2 + 2k\dim(\Gamma) = \dim(\Omega)^2 + 2\dim(\Omega).\end{aligned}$$

Thus, it remains to prove the result under the assumption that $\mathrm{Soc}_d(G)$ is non-abelian and definably simple. In this case, by Theorem 1.1, (G, Ω) is semialgebraic in a definable real closed field R, and by transfer we may suppose that this field is $\mathbf{R}$. We thank Linus Kramer for contributing and patiently explaining the argument below.

Let H be the stabiliser of some $\alpha \in \Omega$ (thus $\dim \Omega = \dim G - \dim H$) and let $L(G), L(H)$ be the Lie algebras of G, H respectively. Let $L(G)^{\mathbf{C}}$, $L(H)^{\mathbf{C}}$ be their complexifications and let K be a maximal complex algebraic subalgebra of $L(G)^{\mathbf{C}}$ containing $L(H)^{\mathbf{C}}$. Certainly, $L(G)^{\mathbf{C}}$ is either simple or semi-simple (in the case where $L(G)$ was already a complex Lie algebra, then $L(G)^{\mathbf{C}}$ is a direct sum of two isomorphic copies of the same simple Lie algebra) and K is reductive or maximal parabolic by [**7**, 30.4, p. 187] applied to the normaliser of K in the algebraic group belonging to $L(G)^{\mathbf{C}}$ (where we call the Lie algebra K parabolic if the corresponding subgroup of $L(G)^{\mathbf{C}}$ is). Note that K is reductive if and only if the connected component of the corresponding subgroup of $L(G)^{\mathbf{C}}$ is reductive as an algebraic group.

If K is parabolic, then the bound holds as can be seen using [**29**, p. 36, Proposition 4.3.5] together with the formulae in [**8**, Table 2.2, p. 44], for let $\tilde{G}$ and $\tilde{K}$ be the corresponding linear algebraic groups, so $\dim_{\mathbf{C}} \tilde{G} = \dim_{\mathbf{R}} G$ and $\dim_{\mathbf{C}} \tilde{K} \geqslant \dim_{\mathbf{R}} H$. The groups $\tilde{G}, \tilde{K}$ are connected. Now we have $\dim_{\mathbf{C}} \tilde{G}/\tilde{K} = \dim_{\mathbf{C}} \tilde{G} - \dim_{\mathbf{C}} \tilde{K} = \ell(w_I) = \ell(w_0) - \ell(w_0^I) \leqslant \dim \Omega$ where W and W_I are Weyl groups belonging to $\tilde{G}$ and $\tilde{K}$, respectively and w_0 and w_0^I are the longest word in W and W_I, respectively, and w_I is the shortest representative of the coset $w_0 W_I \in W/W_I$. The coxeter diagram of W_I is obtained from the coxeter diagram of W by removing one node (see [**29**, p. 35, Theorem 4.3.4]) because $\tilde{K}$ is maximal parabolic. To obtain a lower bound for $\dim \Omega$ it thus suffices to find a maximal parabolic subgroup P of $\tilde{G}$ such that the shortest representative of the coset $w_0 W_I$ is as short as possible.

For example, if $\tilde{G}$ is of type A_n, then $\ell(w_0) = \frac{1}{2}n(n+1)$ and by going over the possibilities, we see that $\ell(w_I)$ is maximal if the parabolic subgroup is of type A_{n-1}, so $\ell(w_I) = \frac{1}{2}(n-1)n$. Hence $\dim \Omega = \frac{1}{2}(n^2+n-n^2+n) = n$, that is, $\tilde{G} \cong \mathrm{PSL}(n+1, \mathbf{C})$ is acting on the n-dimensional *complex* projective space, and G is a finite central extension of $\mathrm{PSL}(n+1, \mathbf{R})$ acting on the n-dimensional *real* projective space. Hence $\dim G = n^2 + 2n$ and the bound is sharp.

If the Lie algebra is the direct sum of two isomorphic simple complex Lie algebras, then the root system of $\tilde{G}$ is reducible. To obtain the coxeter diagram of a maximal parabolic subgroup, one node is removed in one of the connected components of the diagram of $\tilde{G}$. The same calculations as before apply, the unchanged part of the diagram cancels out and we obtain the same bound.

Finally, suppose that K (as well as $L(G)^{\mathbf{C}}$) is reductive. By [**15**, p. 247, Theorem 12], there are compact real forms of $L(G)^{\mathbf{C}}$ and K with corresponding compact Lie groups $\bar{G}$ and $\bar{K}$. Note that by definition of the real form, we have $\dim_{\mathbf{R}} \bar{G} = \dim_{\mathbf{C}} \tilde{G} = \dim_{\mathbf{R}} G$ and $\dim_{\mathbf{R}} \bar{K} = \dim_{\mathbf{C}} K \geqslant \dim_{\mathbf{R}} H$.

Then, putting together [**15**, Theorem 2, Problem 3, Problem 4, Problem 2, p. 239], we see that $\bar{G}$ is a maximal compact subgroup of the complex Lie group $G^{\mathbf{C}}$ belonging to $L(G)^{\mathbf{C}}$ and likewise $\bar{K}$ is a maximal compact subgroup of the complex Lie group belonging to K. Since all maximal compact subgroups are conjugate ([**5**, Chapter VI, 2.2(ii)], note that $G^{\mathbf{C}}$ is semisimple) we may assume that $\bar{K} \leqslant \bar{G}$. Now $\bar{K}$ and $\bar{G}$ are compact Lie groups. If $\bar{G}$ is simple and hence acting faithfully on the right cosets of $\bar{K}$, we can apply [**27**, 96.13] to obtain $\dim \bar{G} \leqslant \frac{1}{2}n'(n'+1)$ with $n' = \dim \bar{G} - \dim \bar{K}$. Since we have $\dim \bar{G} = \dim G$ and $\dim \bar{K} \geqslant \dim H$, the same inequality holds for G and $n = \dim \Omega = \dim G - \dim H$. If $\bar{G}$ is not simple, and hence a direct product of two isomorphic simple groups, then we apply the same inequality to one of the factors and obtain $\frac{1}{2}\dim \bar{G} \leqslant \frac{1}{2}n'(n'+1)$ with $n' = \dim \bar{G} - \dim \bar{K}$, hence $\dim \bar{G} \leqslant n'(n'+1)$. Thus, as before we conclude that $\dim G \leqslant n(n+1)$ where $n = \dim \Omega = \dim G - \dim H$.

A possible further application of Theorems 1.1 and 1.2 would be to classify definably primitive permutation groups (G, Ω) in o-minimal structures such that G has finitely many orbits on Ω^2 (this is a generalisation of the classification of doubly transitive groups in [**32**]). Certainly, any point stabiliser is infinite, so by Theorem 1.1 we can assume that (G, Ω) is semialgebraic over $\mathbf{R}$. If G has abelian definable socle, the point stabiliser will be an irreducible subgroup of $\mathrm{GL}(m, R)$ (its connected component a central product of finitely many definably simple groups) with finitely many orbits on vectors. With $\mathbf{C}$ in place of R, such groups are classified in [**9**]. Also, connected irreducible closed subgroups of $\mathrm{GL}(m, F)$ with finitely many orbits on 1-spaces have been classified in [**4**] for *any* algebraically closed field F, using different methods. In the case when G has non-abelian definable socle, if H is the point stabiliser then an easy dimension argument gives that $2\dim(L(H)) \geqslant \dim(L(G))$.

Actions of type 3 in Theorem 1.2 violate this, so G will be essentially a wreath product (in the product action) of definably simple semialgebraic groups over $\mathbf{R}$, and classification should be feasible.

We mention three further problems.

Problem 9.3. Describe definable infinite permutation groups with finite point stabiliser which are (abstractly) primitive.

Problem 9.4. Describe definable permutation groups with non-abelian minimal normal subgroup which are definably primitive but not (abstractly) primitive.

Problem 9.5. In the abelian socle case of Theorem 1.1, under the assumption that the point stabiliser is infinite, do we need connectedness to ensure that the permutation group is semialgebraic?

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Note added in proof, September 2000. It follows easily from Theorem 1.1 and the proof of Proposition 4.1 that any 2-transitive infinite permutation group definable in an o-minimal structure is definable isomorphic to an R-semialgebraic permutation group for some definable real closed field R. In this case, at the end of the proof of Proposition 4.1, we find that $U = A$, for U has finitely many L-translates, and by 2-transitivity every element of A must lie in an L-translate of U, that is, these finitely many translates cover A. It follows that $L^o = H$ is definably isomorphic to an R-semialgebraic group. As $L \leqslant \mathrm{GL}(A)$ and $|L : L^o|$ is finite, the same holds for L (and hence for G).

References

1. A. Borel, *Linear algebraic groups* (Springer, Berlin, 1991).
2. V. V. Gorbatsevich, A. L. Onishchik and E. B. Vinberg, *Foundations of Lie theory and Lie transformation groups* (Springer, Berlin, 1997).
3. U. Gropp, 'Action d'un groupe ω-stable sur un ensemble de rang de Morley 2', Thèse de doctorat, Université Paris VII, 1991.
4. R. M. Guralnick, M. W. Liebeck, H. D. Macpherson and G. M. Seitz, 'Modules for algebraic groups with finitely many orbits on subspaces', *J. Algebra* 196 (1997) 211–250.
5. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces* (Academic Press, San Diego, 1978).
6. E. Hrushovski, 'Almost orthogonal regular types', *Ann. Pure Appl. Logic* 45 (1989) 139–155.
7. J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics (Springer, Berlin, 1987).
8. J. E. Humphreys, *Reflections groups and coxeter groups* (Cambridge University Press, Cambridge, 1990).
9. V. Kac, 'Some remarks on nilpotent orbits', *J. Algebra* 64 (1980) 190–213.
10. M. W. Liebeck, C. E. Praeger and J. Saxl, 'On the O'Nan–Scott theorem for finite primitive permutation groups', *J. Austral. Math. Soc.* 44 (1988) 389–396.
11. H. D. Macpherson and A. Pillay, 'Primitive permutation groups of finite Morley rank', *Proc. London Math. Soc.* (3) 70 (1995) 481–504.
12. H. D. Macpherson and C. E. Praeger, 'Infinitary versions of the O'Nan–Scott theorem', *Proc. London Math. Soc.* (3) 68 (1994) 518–540.
13. A. Mosley, 'Groups definable in topological structures', PhD Thesis, Queen Mary and Westfield College, 1996.

14. A. Nesin, A. Pillay and V. Razenj, 'Groups of dimension two and three over o-minimal structures', *Ann. Pure Appl. Logic* 53 (1991) 279–296.
15. A. L. Onishchik and E. B. Vinberg, *Lie groups and algebraic groups* (Springer, Berlin, 1990).
16. M. Otero, Y. Peterzil and A. Pillay, 'Groups and rings definable in o-minimal expansions of real closed fields', *Bull. London Math. Soc*. 28 (1996) 7–14.
17. Y. Peterzil and S. Starchenko, 'A trichotomy theorem for o-minimal structures', *Proc. London Math. Soc*. (3) 77 (1998) 481–523.
18. Y. Peterzil and C. Steinhorn, 'Definable compactness and definable subgroups of o-minimal groups', *Bull. London Math. Soc*. 28 (1996) 7–14.
19. Y. Peterzil, A. Pillay and S. Starchenko, 'Definably simple groups in o-minimal structures', *Trans. Amer. Math. Soc*. (to appear).
20. Y. Peterzil, A. Pillay and S. Starchenko, 'Simple algebraic and semialgebraic groups over real closed fields', *Trans. Amer. Math. Soc*. (to appear).
21. Y. Peterzil. A. Pillay and S. Starchenko, 'Simple groups definable in o-minimal structures', *Logic Colloquium* '96*: Proceedings of the colloquium held in San Sebastian, Spain, July* 9–15*,* 1996 (ed. J. M. Larrazabel, D. Lascar and G. Mints, Springer, Berlin, 1998) 211–218.
22. Y. Peterzil, A. Pillay and S. Starchenko, 'Linear groups definable in o-minimal structures', preprint.
23. A. Pillay, 'On groups and fields definable in o-minimal structures', *J. Pure Appl. Algebra* 53 (1988) 239–255.
24. A. Pillay, 'An application of model theory to real and *p*-adic algebraic groups', *J. Algebra* 126 (1989) 139–146.
25. A. Pillay and C. Steinhorn, 'Definable sets in ordered structures I', *Trans. Amer. Math. Soc*. 295 (1986) 565–592.
26. V. Razenj, 'One-dimensional groups over an o-minimal structure', *Ann. Pure Appl. Logic* 53 (1991) 269–277.
27. H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen and M. Stroppel, *Compact projective planes* (de Gruyter, Berlin, 1995).
28. L. L. Scott, 'Representations in characteristic *p*', *Santa Cruz Conference on Finite Groups*, Proceedings of Symposia in Pure Mathematics 37 (American Mathematical Society, Providence, RI, 1980) 318–331.
29. T. A. Springer, 'Linear algebraic groups', *Algebraic geometry IV* (Springer, Berlin, 1994) 1–121.
30. R. Steinberg, 'Lectures on Chevalley groups', Yale University lecture notes, 1968.
31. D. Suprunenko, 'Soluble and nilpotent linear groups', Translations of Mathematical Monographs 9 (American Mathematical Society, Providence, RI, 1963).
32. K. Tent, 'Sharply *n*-transitive groups in o-minimal structures', *Forum Math*. 12 (2000) 65–75.

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