ON GROUPS AND RINGS DEFINABLE IN O-MINIMAL EXPANSIONS OF REAL CLOSED FIELDS

MARGARITA OTERO, YA'ACOV PETERZIL AND ANAND PILLAY

ABSTRACT

Let $\langle R, <, +, \cdot \rangle$ be a real closed field, and let $\mathcal{M}$ be an o-minimal expansion of R. We prove here several results regarding rings and groups which are definable in $\mathcal{M}$. We show that every $\mathcal{M}$-definable ring without zero divisors is definably isomorphic to R, $R(\surd(-1))$ or the ring of quaternions over R. One corollary is that no model of T_{exp} is interpretable in a model of T_{an}.

1. *Introduction*

Our main interest here is in rings which are definable within o-minimal structures. Basic results on the subject were obtained in [5]; for example, it was shown there that every definable group or field can be definably equipped with a 'manifold' structure. It followed that every definable field must be either real closed, in which case it is of dimension 1, or algebraically closed of dimension greater than 1. In [4] the particular nature of semialgebraic sets (that is, sets definable in $\langle \mathbb{R}, <, +, \cdot \rangle$) was used to show that every semialgebraic real closed field is semialgebraically isomorphic to the field of reals.

Some rich mathematical structures have been recently shown to be o-minimal (for example, $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$; see [9]). In this paper we look at arbitrary o-minimal expansions of real closed fields, and investigate the groups and rings definable there. We show that every definably connected definable ring with a trivial annihilator is definably isomorphic to a subalgebra of the ring of matrices over R. We then conclude the following.

THEOREM 1.1. *If K is an $\mathcal{M}$-definable ring with no zero divisors, then K is definably isomorphic to R, to $R(\surd(-1))$ (the R-version of the complex field) or to the ring of quaternions over R.*

Along the way, we prove a definable analogue, in any o-minimal expansion of a real closed field, of a theorem on uniqueness of solutions to differential equations over the reals (see Theorem 2.3). As a corollary to Theorem 1.1, we show that no model which is elementarily equivalent to $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$ can be interpreted in a model of T_{an}, the theory of the field of reals expanded by restricted analytic functions (see Corollary 4.5).

It is of interest to compare Theorem 1.1 to the corresponding situation in the strongly minimal context. (A structure $\mathcal{M}$ is called *strongly minimal* if every definable subset of every elementary extension of $\mathcal{M}$ is finite or cofinite.) As in the semialgebraic case, it is known (see [6]) that every algebraic field (that is, a field definable inside

Received 25 May 1993; revised 12 January 1995.

1991 *Mathematics Subject Classification* 12L12.

First author partially supported by a CICYT PB890379C0202 grant.

$\langle \mathbb{C}, +, \cdot \rangle$) is algebraically isomorphic to the complex field itself. However, what follows from the work of Hrushovski on strongly minimal structures (see [3]) is that given any two algebraically closed fields F_1, F_2, there is a strongly minimal structure which expands both of them. Moreover, the two fields can be put together in an 'orthogonal' way, and in particular there will be no definable isomorphism in the expansion between the two fields. Theorem 1.1 gives an opposite result in the o-minimal context, and, as the proof shows, if $\mathcal{M}$ is an o-minimal expansion of a real closed field R, then definable objects in $\mathcal{M}$ are very strongly related to the structure of R. Indeed, at the heart of the proof is the fact that every definable function in $\mathcal{M}$ is almost everywhere differentiable with respect to the field R.

Other related work appears in [10] and [8], where an abstract characterization of definable rings in strongly minimal (actually, in ω_1-categorical) structures is given.

The first two authors began working on this paper during their visit at the University of Illinois at Chicago in 1992. They thank Dave Marker, Laura Mayer and John Baldwin for organizing the Model Theory year there.

We begin with some preliminaries. We say that a group (a field) is definable inside a structure $\mathcal{M}$ if the underlying set is definable in $\mathcal{M}$ and the graph of the group (field) operations are definable in $\mathcal{M}$. We always take 'definable' to mean 'definable with parameters'. We remind the reader that a linearly ordered structure $\mathcal{M} = \langle M, <, \ldots \rangle$ is called *o-minimal* if every definable subset of M is a finite union of intervals whose endpoints lie in $M \cup \{\pm\infty\}$. We are interested here only in densely ordered structures without endpoints. For basic results on o-minimal structures, see [2].

DEFINITION 1.2. Let $\mathcal{M}$ and $\mathcal{N}$ be first-order structures. We say that $\mathcal{N}$ *is interpretable in* $\mathcal{M}$ if the following holds. There is an $\mathcal{M}$-definable set $\hat{N} \subseteq M^k$ for some k, and a definable equivalence relation E on $\hat{N}$ such that for every atomic n-ary relation in $\mathcal{N}$, there is an $\mathcal{M}$-definable relation on $(\hat{N}/E)^n$, and for every atomic n-ary function in $\mathcal{N}$, there is an $\mathcal{M}$-definable function from $(\hat{N}/E)^n$ into $\hat{N}/E$ such that the structure $\mathcal{N}$ is isomorphic to the structure these relations and functions induce on $\hat{N}/E$.

As we know (see [2]), if $\mathcal{M}$ is an o-minimal expansion of an ordered group, then every structure of the form $\hat{N}/E$ above is definably isomorphic to a structure which is *definable* in $\mathcal{M}$, namely, we may assume that the equivalence relation E is just equality. Thus every characterization that we give below of $\mathcal{M}$-definable groups or rings holds also for the corresponding interpretable objects.

Unless otherwise stated, we shall assume from now on that $\langle R, <, +, \cdot \rangle$ *is a real closed field and* $\mathcal{M} = \langle R, <, +, \cdot, \ldots \rangle$ *is an o-minimal expansion of it.* All definable objects are assumed to be definable in $\mathcal{M}$. Since R is an ordered field, the usual definition of derivative makes sense there. We always use 'differentiable' in this sense.

We make extensive use of the following fact (see [2]).

LEMMA 1.3. *If* $F\colon R^m \to R^n$ *is a definable function, and* k *is a natural number, then there is a definable open and dense set* $D \subseteq R^m$, *such that* F *is* k-*differentiable on* D.

2. *Uniqueness of solutions to differential equations*

Over the reals, there are some well-known existence and uniqueness theorems for differential equations. We prove here a definable analogue, for o-minimal expansions of real closed fields, of such a uniqueness theorem. Clearly, an existence theorem for definable solutions to such equations fails under any reasonable assumptions. The uniqueness theorem for functions of one variable is rather easy to state and prove, but as we prove here a multi-variable version, we need some earlier results on o-minimal structures.

As is shown in [2], if $\phi: R^m \to R$ is a definable continuous function and $D \subseteq R^m$ is a definable closed and bounded set, then ϕ is bounded on D and attains a maximum there. For $\bar{x} \in R^m$, we define the norm of x to be $|\bar{x}| = \max\{|x_i|: 1 \leqslant i \leqslant m\}$, and use from now on x instead of $\bar{x}$. For T a linear transformation from R^m into R^n, we put the norm of T, $|T|$, to be

$$|T| = \max\{|Tx|: |x| \leqslant 1, x \in R^m\}.$$

For such T, let $[T]$ be the $n \times m$ matrix which represents it over the standard basis. The norm of T as a linear transformation is at most $m \times$ (the norm of $[T]$ as a vector in R^{nm}).

For $U \subseteq R^m$ an open set, if $\phi: U \to R^n$ is a differentiable function (that is, all partial derivatives of ϕ with respect to R exist and are continuous), we denote by $d_x(\phi)$ the linear transformation given by the $n \times m$ matrix of partial derivatives of ϕ evaluated at x.

LEMMA 2.1 [2]. *If $U \subseteq R^m$ is an open rectangular box, $a \in U$, and $\phi: U \to R^n$ is differentiable on U, then for every $x \in U$ we have*

$$|\phi(x) - \phi(a)| \leqslant |x-a| \max\{|d_y \phi|: |y-a| \leqslant |x-a|\}.$$

From now on we always think of $d_y \phi$ as an $n \times m$ matrix and take $|d_y \phi|$ to be the norm of the corresponding vector in R^{nm}. The above lemma can be adjusted through multiplication by a constant on either side of the inequality.

LEMMA 2.2. *Let ϕ, U and a be as in the above lemma, $\phi(a) = 0$, and assume that there is $C \in R^+$ such that for all $x \in U$ we have $|d_x \phi| \leqslant C|\phi(x)|$. Then $\phi(x) = 0$ for all $x \in U$.*

Proof. To simplify notation, we shall assume that $a = 0$. We shall show first that there is some neighbourhood of 0 on which ϕ vanishes. Assume that this fails, so in particular there is no neighbourhood of 0 on which $d_x \phi$ vanishes. Let $S = \{x \in U: \max\{|d_y \phi|: |y| \leqslant |x|\} = |d_x \phi|\}$, namely, S contains all points x which realize the maximum of $d_{(-)} \phi$ on the rectangle determined by x. By the fact that such a maximum is indeed realized inside every closed and bounded rectangle, it is easy to see that 0 is a limit point of S. By Lemma 2.1, there is a nonzero c' such that, given any $x \in S$, we have $|\phi(x)| \leqslant |x|\,|d_x \phi|\, c'$. Taken together with our assumption on $d_x \phi$, we obtain

$$|d_x \phi| \leqslant C'|x|\,|d_x \phi| \quad \text{for some nonzero } C' \text{ independent of } x.$$

But if we pick $x \in S$ close enough to 0 so that $|x|\, C' < 1$, we must have $|d_x \phi| = 0$, which implies that $d_y \phi = 0$ for all y in the box $\{y \in R^m: |y| \leqslant |x|\}$. Contradiction.

To show that ϕ vanishes everywhere, just notice that $Z = \{x \in U: \phi(x) = 0\}$ is a closed set (clearly) and open (by applying the above argument to any point of Z). But U is definably connected (that is, has no definable clopen subset), so $Z = U$.

THEOREM 2.3. *Let $F(x,y)$ be a definable function, defined on a closed rectangular box $U = I_1 \times I_2$ in R^{m+n}, which for every (x,y) gives an $n \times m$ matrix over R. Assume that F is differentiable (as a function from U into R^{nm}). For $(a,b) \in \operatorname{Int}(U)$, consider the system of equations given by*

$$d_x(\phi) = F(x, \phi(x)), \tag{1}$$

$$\phi(a) = b. \tag{2}$$

If $\phi_1, \phi_2: I_1 \to I_2$ are two definable solutions to the system, then there is an open neighbourhood $V \subset I_1$ of a such that $\phi_1 = \phi_2$ on V.

Proof. For $x \in I_1$ let F_x be the function on I_2 given by $F_x(y) = F(x,y)$. Since $d_y(F_x)$ is continuous in (x,y), there is $C \in R$ such that $C \geqslant |d_y(F_x)|$ for all $(x,y) \in U$. By Lemma 2.1, if $(x,y_1),(x,y_2) \in \operatorname{Int}(U)$, then $|F(x,y_1) - F(x,y_2)| \leqslant C|y_1 - y_2|$.

Assume that ϕ_1, ϕ_2 are both solutions to the system. Let $V \subset I_1$ be an open rectangular box which contains a, such that for every $x \in V$ we have $\phi_1(x), \phi_2(x) \in I_2$. Given $x \in V$, we have then

$$|d_x(\phi_1 - \phi_2)| = |F(x, \phi_1(x)) - F(x, \phi_2(x))| \leqslant C|\phi_1(x) - \phi_2(x)|.$$

We apply now Lemma 2.2 to the function $\phi_1 - \phi_2$, and so ϕ_1 must equal ϕ_2 on V.

REMARK. By Lemma 1.3, given any definable $F: V \subseteq R^{m+n} \to R^{mn}$, there is an open $U \subseteq V$ on which F is differentiable, and hence the assumptions of the above theorem hold there.

3. *Embedding centreless groups in* $\mathrm{GL}_n(R)$

For $\langle G, * \rangle$ an $\mathcal{M}$-definable group, we say that G admits a definable (k-differentiable) group-manifold structure over $\mathcal{M}$ if there are open definable sets $U_1, \ldots, U_r \subseteq R^n$ for some n and definable injective maps $\pi_i: U_i \to G$, such that $G = \bigcup \pi_i(U_i)$ and the transition maps are continuous (k-differentiable) with respect to the underlying real closed field structure. Moreover, the maps $(\;)^{-1}, *$ on G are continuous (k-differentiable) when read through the charts. We similarly define the notion of a ring-manifold, now ensuring that the additive group operation, its inverse and the multiplication function are (continuous) k-differentiable. The following is proved in Proposition 2.5, Remark 2.6 and Proposition 3.1 of [5].

PROPOSITION 3.1. *Every definable group or field in an o-minimal structure admits a definable (k-differentiable) manifold.*

Actually, one can show that the manifold structure in the proposition is unique in the sense that given any two such manifold structures, the identity map gives a homeomorphism between the two.

LEMMA 3.2. *Let $\langle G, e, * \rangle$ be a definable group equipped with a differentiable group-manifold G, definably connected with respect to the manifold topology. Let σ, τ be $\mathcal{M}$-definable group homomorphisms of G. Then*

(i) *the maps* σ, τ *are differentiable everywhere on* G;
(ii) *if* $\sigma \neq \tau$, *then* $d_e(\sigma) \neq d_e(\tau)$.

Proof. (i) By Lemma 1.3, every definable map on G is differentiable, with respect to the manifold structure, on an open and dense subset of G. Assume that σ is differentiable on a neighbourhood of some point h. For $a \in G$, denote by l_a the function on G given by $x \mapsto a * x$. We can factor σ as

$$\sigma(x) = \sigma(h^{-1}) * \sigma(h * x) = l_{\sigma(h^{-1})}(\sigma(l_h(x))).$$

As was shown in [2], basic calculus is still true for definable maps. In particular, the multi-variable chain rule holds for such maps, so the differentiability of left-multiplication everywhere and of σ at h implies the differentiability of σ at e. We can show similarly the differentiability of σ on all of G.

(ii) Given g, we can factor σ as the composition

$$\sigma(x) = \sigma(g) * \sigma(g^{-1} * x) = l_{\sigma(g)}(\sigma(l_{g^{-1}}(x))).$$

We fix now (U, π) in the manifold chart such that, without loss of generality, $e \in U \subseteq G$ and $\pi = \mathrm{id}$. Given $h \in U$, h close enough to e, d_h is well-defined and we have

$$d_h(\sigma) = d_e(l_{\sigma(h)}) \cdot d_e(\sigma) \cdot d_h(l_{h^{-1}}).$$

Let $D = d_e(\sigma)$, an $n \times n$ matrix (say, assuming that G is of dimension n), and define $F(x, y)$ as a (partial) function from $G \times G$ into $R^{n \times n}$ by

$$F(x, y) = d_e(l_y) \cdot D \cdot d_x(l_{x^{-1}}).$$

(For F to be well-defined, we need to restrict it to x and y in a small neighbourhood of e.) F is clearly a differentiable function on a neighbourhood of (e, e), and by the argument above, if $d_e(\tau) = d_e(\sigma)$, then both τ and σ, on a neighbourhood of e, are solutions to the system

$$d_x(\phi) = F(x, \phi(x)),$$
$$\phi(e) = e.$$

By Theorem 2.3, the above system has at most one definable solution, hence on some neighbourhood of e, $\tau = \sigma$. Since G is definably connected, $\tau = \sigma$ everywhere on G.

COROLLARY 3.3. *Let* G *be as above and centreless* ($Z(G) = \{e\}$). *Then* G *can be definably embedded into* $\mathrm{GL}_n(R)$, *the general* $n \times n$ *linear group over the field* R.

Proof. Let n be the dimension of G, and fix U as in the last proof. Consider the (definable) adjoint map $\mathrm{Ad}\colon G \to \mathrm{GL}_n(R)$, given by $\mathrm{Ad}(g) = d_e(\sigma_g)$, where σ_g is the automorphism $G\colon h \mapsto g^{-1}hg$. Ad is a homomorphism, and by Lemma 3.2, the kernel of Ad is $Z(G)$, so by our assumption it is an injective map.

4. *On rings*

We first prove some basic results for any ring definable in an o-minimal structure, omitting the assumption that the structure is an expansion of a real closed field.

LEMMA 4.1. *Let* $\langle K, +, \cdot, 0\rangle$ *be a definable ring in an o-minimal structure* $\mathcal{N}$, *and assume that the group* $\langle K, +\rangle$ *admits a definable group-manifold structure* M. *Then* M *makes* K *into a ring-manifold. If* $\mathcal{N}$ *is an o-minimal expansion of a real closed field and* M *is a k-differentiable group-manifold, then it is also a k-differentiable ring-manifold.*

Proof. We shall show the latter, that is, we assume that $\mathcal{N}$ is an o-minimal expansion of a real closed field and that M induces a k-differentiable group-manifold on $\langle K, +\rangle$.

For $a \in K$, the functions $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are k-differentiable with respect to M (see (i) of Lemma 3.2). Since definable functions on M are k-differentiable almost everywhere with respect to the underlying field of $\mathcal{N}$, such functions on K are k-differentiable almost everywhere with respect to M. In particular, there are open sets $U, V \subseteq K$ such that multiplication in K is k-differentiable on $U \times V$ with respect to M. Fix $(x_0, y_0) \in U \times V$. Then given $(x_1, y_1) \in K^2$, let $x_2 = x_1 - x_0$ and $y_2 = y_1 - y_0$. For every $x, y \in K$, we have then

$$x \cdot y = (x - x_2) \cdot (y - y_2) + x_2 \cdot y + x \cdot y_2 + x_2 \cdot y_2.$$

Each of the summands is a k-differentiable function of (x, y) on a neighbourhood of (x_1, y_1), hence the function $x \cdot y$ is also differentiable there.

COROLLARY 4.2. *If $\langle K, +, \cdot\rangle$ is a definable ring in an o-minimal structure $\mathcal{N}$, then it admits a definable ring-manifold structure. If $\mathcal{N}$ is an o-minimal expansion of a real closed field and k is a natural number, then K admits a definable k-differentiable ring-manifold.*

Proof. This is immediate from Proposition 3.1 and Lemma 4.1.

We now return to the assumption that $\mathcal{M}$ is an o-minimal expansion of a real closed field R.

LEMMA 4.3. *Let $\langle K, \hat{+}, \hat{\cdot}\rangle$ be an $\mathcal{M}$-definable ring of dimension n, definably connected with respect to the manifold topology, such that either the left or right annihilator of K is trivial, that is, $\{a \in K: a \hat{\cdot} x = 0 \text{ for all } x \in K\} = \{0\}$ or $\{a \in K: x \hat{\cdot} a = 0 \text{ for all } x \in K\} = \{0\}$. Then there is an $\mathcal{M}$-definable ring isomorphism between K and a subalgebra of $M_n(R)$, the $n \times n$ ring of matrices over R.*

Proof. By Corollary 4.2, K admits a definable differentiable manifold structure. Denote by e the zero of the ring K. Without loss of generality, there is an open set $U \subseteq \mathbb{R}^n$, $e \in U \subseteq K$, such that $\hat{+}$ and $\hat{\cdot}$ are differentiable on U. Without loss of generality, the left annihilator is trivial.

For $a \in K$ and $x \in K$, define $\lambda_a(x) = a \hat{\cdot} x$. We have then $d_e(\lambda_a(\lambda_b)) = d_e(\lambda_a) \cdot d_e(\lambda_b)$. So, for every $a, b, c \in K$,

$$(a \hat{\cdot} b = c) \quad \Rightarrow \quad (d_e(\lambda_a) \cdot d_e(\lambda_b) = d_e(\lambda_c)),$$

where the multiplication on the right is the matrix multiplication induced from R.

Now consider the map $P(x, y) = x \hat{+} y$. Since P is continuous, it sends a neighbourhood of (e, e) in $U \times U$ into a neighbourhood of e in U. For every $x \in K$, we have $P(e, x) = x = P(x, e)$, so $d_{(e,e)}(P) = (I, I)$, where I is the $n \times n$ identity matrix. Again, using the multi-variable chain rule, we obtain, for every $a, b, c \in K$,

$$(a \hat{+} b = c) \quad \Rightarrow \quad \left(d_{(e,e)}(P) \cdot \begin{pmatrix} d_e(\lambda_a) \\ d_e(\lambda_b) \end{pmatrix} = d_e(\lambda_c)\right) \quad \Leftrightarrow \quad (d_e(\lambda_a) + d_e(\lambda_b) = d_e(\lambda_c)),$$

where the rightmost addition is now the usual matrix addition over the field R.

We showed then that the map $a \mapsto d_e(\lambda_a)$ is a ring homomorphism from K into the ring of matrices $M_n(R)$. In the case that K is a field, this must be an embedding. In the case that K is not a field, our assumptions on K imply that the map $a \mapsto \lambda_a$ is injective, hence by Lemma 3.2, the map $a \mapsto d_e(\lambda_a)$ is also injective. Note that if K has an identity element 1, then the image of K in $M_n(R)$ has the identity matrix $I = d_e(\lambda_1)$ as its identity element.

We may assume, then, that K is a definable subring of $M_n(R)$. For $A \in K$, define $G = \{r \in R: rA \in K\}$. G is an infinite subgroup of $\langle R, + \rangle$, so by o-minimality, $G = R$. But then K is an algebra over R.

The last part of the above proof shows that every definable subring of $M_n(R)$ is actually a finite dimensional algebra over R.

Remark. Since every definable group can be made into a ring by defining a trivial multiplication, we cannot hope to totally omit the assumptions on the annihilator of K.

If K is not definably connected, then one can show that the definable component J of $\hat{0}$ (in the manifold topology) is an ideal in K, and that K/J is finite (see [5] for similar results). If one of the annihilators of the ring J is trivial, then, by the above, J can be definably embedded in $M_n(R)$.

The lemma below uses only the fact that $\langle K, + \rangle$ has the DCC property on definable subgroups, namely that there is no descending chain of definable subgroups. As was shown in [5, Remark 2.13], every definable group in an o-minimal structure has this property.

Lemma 4.4. *If $K = \langle K, +, \cdot, 0 \rangle$ is a definable ring in an o-minimal structure and K has no zero divisors, then it is a division ring (and in particular it has an identity element).*

Proof. Let $K^* = K \setminus \{0\}$. For $a \in K^*$, consider the (injective) map $x \mapsto a \cdot x$. We have $a \cdot K = K$, since otherwise we should obtain a descending chain of subgroups by applying this map repeatedly. So there is $b_1 \in K^*$ such that $a \cdot b_1 = a$. Similarly, there is $b_2 \in K^*$ such that $b_1 \cdot b_2 = b_1$. But then $a \cdot b_2 = a \cdot b_1 \cdot b_2 = a \cdot b_1$. Since a is not a zero divisor, $b_2 = b_1$, and hence $b_1 \cdot b_1 = b_1$.

But now, given any $c \in K^*$, we have $c \cdot b_1 = c \cdot b_1 \cdot b_1$, hence $c \cdot b_1 = c$. Also, $b_1 \cdot c = b_1 \cdot b_1 \cdot c$, hence $b_1 \cdot c = c$. So b_1 is the identity element of K, and since $c \cdot K^* = K^* = K^* \cdot c$ for every $c \in K^*$, the set K^* is a group under multiplication.

We are ready now to prove Theorem 1.1. Let K be an $\mathscr{M}$-definable ring with no zero divisors. By Lemma 4.4, K is a division ring and hence, just like Lemma 3.3 in [5], definably connected. By Lemma 4.3 and its proof, we may assume that K is a division ring of $n \times n$ matrices, with addition and multiplication induced from R, and I as its identity element. Moreover, K is a finite dimensional vector space over R. It then contains a definable copy of R, $\hat{R} = R \cdot I$, as a subfield. If K is a field, it is a finite field extension of $\hat{R}$, and since $\hat{R}$ is a real closed field, K equals $\hat{R}$ or $\hat{R}(\sqrt{(-1)})$, the algebraic closure of $\hat{R}$. If K is a division ring but not a field, then $\hat{R}$ is contained in the centre of the multiplicative group of K, so by Frobenius' Theorem (see [7]) it must be the ring of quaternions over $\hat{R}$. (In particular, it is of dimension 4.)

Theorem 1.1 implies that questions about definable, or more generally interpretable, fields in $\mathcal{M}$ can be reduced to questions about definable expansions of R or $R(\sqrt{(-1)})$. One example is the following.

We let T_{an} be the theory of $\mathcal{M}_{\mathrm{an}}$, the real field expanded by restrictions to $[-1, 1]^n$, $n \in \mathbb{N}$, of all functions which are analytic on some open set containing this interval. It was shown by van den Dries [1] that every function of one variable that is definable in $\mathcal{M}_{\mathrm{an}}$ grows asymptotically like some rational power of x, and hence the function e^x is not definable in $\mathcal{M}_{\mathrm{an}}$. We let $T_{\exp}$ be the theory of the real field expanded by the function e^x, and conclude now a strong version of the above.

COROLLARY 4.5. *No model of $T_{\exp}$ is interpretable in a model of T_{an}.*

Proof. Assume that there are formulas which interpret a real closed field K in a model $\mathcal{M}$ of T_{an}, and a formula $\phi(x,y)$ which defines the graph of the exponential function on K. By Theorem 1.1, there is a definable field isomorphism in $\mathcal{M}$ between K and the underlying real closed field of $\mathcal{M}$. Hence there is in $\mathcal{M}$ a definable order-preserving isomorphism of the additive group and the positive multiplicative group in $\mathcal{M}$. But then such an isomorphism is definable in every model of T_{an}, contradicting the fact that e^x is not definable in $\mathcal{M}_{\mathrm{an}}$.

One can show similarly that if $T = Th(\langle \mathbb{R}, <, +, \cdot, \ldots \rangle)$ and for some irrational α the function x^α is definable in T, then no model of T can be interpreted in a model of T_{an}.

References

1. L. VAN DEN DRIES, 'A generalization of the Tarski–Seidenberg theorem and some nondefinability results', *Bull. Amer. Math. Soc.* 15 (1986) 189–193.
2. L. VAN DEN DRIES, 'Tame topology and o-minimal structures', in preparation, 1991.
3. E. HRUSHOVSKI, 'Strongly minimal expansions of algebraically closed fields', *Israel J. Math.* 79 (1992) 129–151.
4. A. NESIN and A. PILLAY, 'Some model theory for compact Lie groups', *Trans. Amer. Math. Soc.* 326 (1991) 453–463.
5. A. PILLAY, 'On groups and fields definable in o-minimal structures', *J. Pure Appl. Algebra* 53 (1988) 239–255.
6. B. POIZAT, *Groupes stables* (Nur al-Mantiq Val-Ma'rifah, Paris, 1987).
7. L. S. PONTRYAGIN, *Topological groups* (Princeton University Press, 1958).
8. J. REINEKE, 'On ω_1-categorical theories of commutative rings', *Notices Amer. Math. Soc.* 20 (1973).
9. A. WILKIE, 'Model completeness results for expansions of the real field II: the exponential function', preprint.
10. B. ZILBER, 'Rings with $\aleph_1$-categorical theories', *Algebra i Logika* 13 (1974) 168–187.

Department of Mathematics
Universidad Autonoma de Madrid
Spain

Department of Mathematics and
Computer Science
Haifa University
Israel

Department of Mathematics
University of Notre Dame
USA