Journal of Pure Applied Algebra 53 (1988) 239–255
North-Holland

ON GROUPS AND FIELDS DEFINABLE IN *o*-MINIMAL STRUCTURES

Anand PILLAY*
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, U.S.A.

Communicated by A. Blass
Received 11 September 1986

The structure $M=(M,<,R_1,R_2,\dots)$ is *o-minimal* if every definable set $X\subset M$ is a finite union of intervals (a,b) and points. Let G be a group definable in M (i.e. G is a definable subset of M^n and the graph of multiplication is also definable). We show that G can be definably equipped with the structure of a 'manifold' over M in which multiplication and inversion are continuous. In the special case $M=(\mathbb{R},<,+,\cdot)$ our construction gives G the structure of a *Nash group*. These results are also used to show that an infinite field definable in an o-minimal structure is real closed or algebraically closed.

0. Introduction

We say the group G is definable in the o-minimal structure M if G and the graph of the group operation are definable subsets of M^k and M^{3k} respectively, for some $k\geq 1$. We show that such a group G can be definably equipped with topological structure making G a topological group. Moreover, G can be covered by a finite number of open sets, each homeomorphic to an open subset of M^n (suitable n). (In particular if the underlying order of M is $(\mathbb{R},<)$, then G is a Lie group.) The same construction works for a field K definable in M, and we use this to show that such a field must be real closed or algebraically closed.

In Section 1 we fix notation concerning the important notion of *dimension*. In Section 2, we first make some preliminary observations on groups G definable in o-minimal structures. Quite a lot of 'ω-stable group theory' goes through.

In particular for X a 'large' subset of G, finitely many translates of X cover G. We then show that G can be definably made into a topological group. Basically the Weil theory of group chunks applies to our situation. Once this is done we can use connectedness arguments in addition to dimension arguments, and in Section 3 the results on fields are obtained.

M is throughout a model $(M,<,\dots)$ where $<$ is a dense linear order without endpoints, and M is o-minimal, i.e. every definable $X\subset M$ is a finite union of intervals (a,b) (where $a\in M\cup\{-\infty\}$, $b\in M\cup\{+\infty\}$) and points.

M is equipped with the interval topology and M^n with the product topology. Unless otherwise stated, these are the topologies referred to.

* Research supported by NSF grant DMS 8601289.

0022-4049/88/$3.50

We will continually refer to previous works on the subject, mainly [3] and [6]. Without recalling the definition of cells, let me mention some crucial properties.

Fact 0.1. (i) Any A-definable $X \subset M^n$ $(A \subset M)$ is a finite disjoint union of A-definable *cells*.

(ii) Any cell $X \subseteq M^n$ is *definably connected* (i.e. has no proper clopen definable subset).

(iii) For any cell $X \subset M^n$ there is $k \leq n$ such that if π is the projection on k suitable coordinate axes, then $\pi(X)$ is open in M^k and π is a homeomorphism between X and $\pi(X)$.

For $A \subset M$, $\mathrm{dcl}(A) = \mathrm{acl}(A) = \{b \in M\colon b$ is definable from $A\}$.

(iv) So dcl( ) is transitive, and by [8] if $a \in \mathrm{dcl}(b \cup A)$ and $a \not\in \mathrm{dcl}(A)$, then $b \in \mathrm{dcl}(a \cup A)$.

Thus we also have

(v) If $X \subset M^n$ is an A-definable cell and $k \leq n$ is as in (iii), then for suitable $1 \leq i_1 < i_2 \cdots < i_k \leq n$, for any $\bar{a} = (a_1, \ldots, a_n) \in X$, $\bar{a} \in \mathrm{dcl}(\{a_{i_1}, \ldots, a_{i_k}\} \cup A)$. $(i_1, i_2, \ldots, i_k$ are the coordinate axes on which the projection π of (iii) is).

For convenience sake we will assume that M is *very saturated*. All our results, however, hold for arbitrary M.

1. Dimension

Definition 1.1. (i) Let $\bar{a} \in M^n$, $A \subset M$. Then $\dim(\bar{a}/A)$ = least cardinality of a subtuple $\bar{a}'$ of $\bar{a}$ such that $\bar{a} \subset \mathrm{dcl}(A \cup \bar{a}')$.

(ii) Let $p(\bar{x}) \in S_n(A)$. Then $\dim p = \dim(\bar{a}/A)$ for some (any) $\bar{a} \in M^n$ realising p.

Lemma 1.2. (i) $\dim(\bar{a}/A)$ = *the cardinality of any maximal algebraically independent over A, subtuple of $\bar{a}$.*

(ii) $A \subset B \Rightarrow \dim(\bar{a}/A) \geq \dim(\bar{a}/B)$.

(iii) $\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A \cup \bar{b}) + \dim(\bar{b}/A)$.

(iv) $\dim(\bar{a}/\bar{b} \cup A) = \dim(\bar{a}/A$, *iff* $\dim(\bar{b}/A \cup \bar{a}) = \dim(\bar{b}/A)$.

(v) *If $p(\bar{x}) \in S_n(A)$ and $A \subset B$, then there is $p \in S_n(B)$ with $p \subset p'$ and* $\dim p = \dim p'$.

Proof. (i)–(iv) follow by Fact 0.1(iv).

For (v) it is enough by (ii) to find $p' \in S_n(B)$ with $\dim p' \geq \dim p$. Suppose $\dim p = k$. Let $\bar{a}$ realize p. Without loss of generality $a_1, \ldots, a_k$ are algebraically independent over A. By the saturation of M we can find inductively $a'_1, \ldots, a'_k \in M$, algebraically independent over B with $\mathrm{tp}(a'_1, \ldots, a'_k/A) = \mathrm{tp}(a_1, \ldots, a_k/A)$. This is clearly sufficient. $\square$

We will say $\bar{a}$ and $\bar{b}$ are independent over A if the equivalent conditions of Lemma 1.2(iv) are satisfied.

Definition 1.3. Let $X \subset M^n$ be A-definable ($A \subset M$). Then $\dim X = \max\{\dim(\bar{a}/A)\colon \bar{a} \in X\}$ ($=\max\{\dim p\colon p \in S_n(A)$, and p is realized in $X\}$).

Note. (i) By Lemma 1.2(v), $\dim X$ does *not* depend on A.

(ii) Let X be A-definable and $\bar{a} \in X$. We will say that $\bar{a}$ *is a generic point of* X *over* A if $\dim(\bar{a}/A) = \dim X$.

Lemma 1.4. *Let $X \subset M^n$ be definable. Then (for $k \leq n$) $\dim X \geq k$ iff some projection of X onto M^k has interior in M^k.*

Proof. Suppose X is A-definable and $\dim X \geq k$ (so $k \leq n$). Let $\bar{a} = (a_1, \ldots, a_n)$ be a generic point of X over A.

Without loss of generality, $a_1, \ldots, a_k$ are algebraically independent over A. Let Y be the projection of X onto the first k coordinate axes. So $Y \subseteq M^k$ is A-definable and contains the point $(a_1, \ldots, a_k)$. By Lemma 1.4 of [6], Y has interior in M^k.

Conversely, suppose some projection Y of X onto M^k has interior. Let X (and so Y) be A-definable. Now Y contains an open B-definable box (i.e., product of intervals) Z in M^k ($B \supset A$). By saturation of M it is easy to find inductively $a_1, \ldots, a_k$ such that $(a_1, \ldots, a_k) \in Z$ and $a_1, \ldots, a_k$ are algebraically independent over B. So $\dim(a_1, \ldots, a_k/B) = k$, and if $\bar{a}$ is a point in X extending $(a_1 \cdots a_k)$ we see that $\dim(\bar{a}/B) \geq k$. So $\dim X \geq k$. $\square$

The following is an immediate consequence of the definition of $\dim X$:

Lemma 1.5. (i) *Let $X \subseteq M^n$ be definable and let $f\colon X \to Y$ be a definable bijection, where $Y \subset M^k$. Then* $\dim X = \dim Y$.

(ii) *If $X_1, \ldots, X_r$ are definable subsets of M^n, then* $\dim(\bigcup_i X_i) = \max\{\dim X_i : i < r\}$.

Proof. (i) Let A be a set over which X and f are defined. Let $\bar{a} \in X$. So $(\bar{a}, f(a)) \subseteq \mathrm{dcl}(\bar{a} \cup A)$ and $(\bar{a}, f(\bar{a})) \subseteq \mathrm{dcl}(f(\bar{a}) \cup A)$. Thus $\dim(\bar{a}/A) = \dim(f(\bar{a})/A)$.

(ii) Immediate. $\square$

Lemma 1.6. *Let $\vartheta(x_1 \cdots x_n, \bar{y})$ be a formula (without parameters say) and for any $\bar{b}$ let $X_{\bar{b}}$ be the subset of M^n defined by $\vartheta(\bar{x}, \bar{b})$. Then for any $k \leq n$ there is a formula $\psi_k(\bar{y})$ without parameters such that for any $\bar{b}$,* $\dim X_{\bar{b}} = k$ *iff $\psi_k(\bar{b})$.*

Proof. An immediate consequence of Lemma 1.4. $\square$

Lemma 1.7. *Let $X \subseteq M^n$ be definable. Then* $\dim X = n$ *iff X has interior in M^n.*

Proof. Again, this follows by Lemma 1.4. $\square$

The following characterization of dimension in terms of definable equivalence relations will be useful when studying definable groups:

Proposition 1.8. $\dim X \geq k+1$ *iff there is a definable equivalence relation E on X infinitely many classes of which have dimension* $\geq k$.

Proof. If $\dim X \geq k+1$, then by Lemma 1.4 some projection of X onto M^k contains an open box B. Define an equivalence relation on B by $\bar{x} \sim \bar{y}$ iff $\bar{x}$ and $\bar{y}$ have the same first coordinate. Lifting this equivalence relation to X gives us the required E.

Conversely, assume the right-hand side holds. By Fact 0.1(i) and Lemma 1.5(ii) we may assume that X is a cell. Suppose by way of contradiction that $\dim X = k$. Then clearly k is as in Fact 0.1(iii). Let E^1 be the equivalence relation on $\pi(X)$ induced by E.

Now let C be a class of E with $\dim C \geq k$. So clearly $\dim C = k$. By Lemma 1.5, $\dim \pi(C) = k$, and so by Lemma 1.7, $\pi(C)$ has interior in M^k. Thus, infinitely many classes of E^1 have interior in M^k. This contradicts [6, Proposition 2.1]. □

We now give a rather more difficult characterization of dimension, which, although not strictly required for the main line of this paper, is interesting in its own right and in terms of the axiomatic notion of dimension given in [7].

Proposition 1.9. *Let* $X \subseteq M^n$ *be definable. Then* $\dim X \geq k+1$ *if and only if there is a definable* $Y \subseteq X$ *such that* $\dim Y \geq k$ *and Y has no interior in X.*

Proof. The left to right direction is easy and is proved as in the left to right direction of Proposition 1.8.

Conversely, suppose that $Y \subset X$, Y has no interior in X and $\dim Y \geq k$. We may assume that $\dim Y = k$ and Y is a cell. We may also assume, without loss of generality, that Y is homeomorphic to $\pi(Y)$, the projection of Y on the first k coordinate axes, and $\pi(Y)$ is open in M^k. Let $\bar{a} = (a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$ be a generic point of Y over A (where X, Y are A-definable). So $a_1, \ldots, a_k$ are independent over A.

Now for each $j = k+1, \ldots, n$, let $\psi_j(a_1, \ldots, a_k, x_j)$ be the formula

$$\exists x_{k+1} \cdots x_{j-1} x_{j+1} \cdots x_n \, [(a_1, a_2, \ldots, a_k, x_{k+1}, \ldots, x_n) \in X] \, .$$

Claim. *For some j,* $\psi_j(a_1, \ldots, a_k, x_j)$ *is infinite.*

If the claim holds, then we can choose b_j satisfying $\psi_j(a_1, \ldots, a_k, x_j)$ with $b_j \not\in \mathrm{dcl}(a_1 \cdots a_k \cup A)$. This clearly gives rise to a point $\bar{c} = (c_1, \ldots, c_n)$ with $c_1 = a_1, \ldots, c_k = a_k$, $c_j = b_j$, which is in X and such that $\dim(\bar{c}/A) \geq k+1$. So $\dim X \geq k+1$ as required.

Proof of Claim. If not, then, as $\dim(a_1 \cdots a_k/A) = k$, then there is, by [3] an open box $Z \subset \pi(Y)$ containing $(a_1, \ldots, a_k)$ such that for all $(x_1, \ldots, x_k) \in Z$, $\psi_j(x_1, \ldots, x_k, x_j)$ is finite, for all j.

Now for $(x_1, \ldots, x_k) \in Z$ and $k + 1 \leq j \leq n$, let $g_j(x_1, \ldots, x_k)$ = the jth coordinate of $\pi^{-1}(x_1, \ldots, x_k)$, and define $f_j^1(x_1, \ldots, x_k)$ = greatest $x_j < g_j(x_1, \ldots, x_k)$ such that $\psi_j(x_1, \ldots, x_k, x_j)$. $f_j^2(x_1, \ldots, x_k)$ = least $x_j > g_j(x_1, \ldots, x_k)$ such that $\psi_j(x_1, \ldots, x_k, x_j)$. Again using [3] and the fact that $\dim(a_1 \cdots a_k/A) = k$ (so any A-definable cell in M^k containing $(a_1, \ldots, a_k)$ must be open), we may assume that for $j = k + 1, \ldots, n$, the functions g_j, f_j^1, f_j^2 are continuous on Z.

Then clearly we can find an open box W in M^n containing $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n)$ such that for any $x_1, \ldots, x_k$, $(x_1, \ldots, x_k, f_{k+1}^i(x_1, \ldots, x_k), \ldots, f_n^i(x_1, \ldots, x_k)) \in W$ for $i = 1, 2$. Then clearly $X \cap W = Y$ and so Y has interior in X, contradicting our assumption and proving the claim, and the proposition. □□

Remark 1.10. Let X be a set, equipped with a topology t. Suppose that X is covered by open subsets $U_1, \ldots, U_s$. Suppose, moreover, that for each $i = 1, \ldots, s$ there is a homeomorphsim π_i between U_i and V_i where V_i is a definable subset of M^{n_i} with its induced topology, and that for each i, j the induced homeomorphism between $\pi_i(U_i \cap U_j) \subset V_i$ and $\pi_j(U_i \cap U_j) \subset V_j$ is definable. Equip X with all the definable structure induced from M by the π_i's. So X is a kind of 'manifold' over M, and satisfies the following:

(i) Every definable subset of X is a Boolean combination of closed definable sets;

(ii) For every definable $Y \subset X$, $\dim Y$ is defined and satisfies $\dim Y \geq k + 1$ iff there is definable $Z \subset Y$ with no interior in Y and $\dim Z \geq k$;

(iii) Every definable $Y \subset X$ is a finite disjoint union of definably connected definable sets;

(iv) The topology on X is explicitly definable.

(i), (iii) and (iv) are easy and (ii) follows from Proposition 1.9. So X with its topology and all its definable structure is an example of what we called a topologically totally transcendental structure in [7].

Finally, in this section we introduce *large* sets.

Definition 1.11. Let $Y \subset X \subset M^n$ be definable. We say that Y is *large* in X if $\dim(X - Y) < \dim X$.

The following is an easy consequence of the definitions:

Lemma 1.12. *Let $Y \subset X$ be definable. Then Y is large in X if and only if for every A over which X and Y are defined, every generic point $\bar{a}$ of X over A is in Y.* □

Proposition 1.13. *Let $X \subset M^n$ be A-definable. Let $\phi(x_1, \ldots, x_n, \bar{y})$ be a formula over $\emptyset$. Then $\{\bar{b}: \phi(\bar{x}, \bar{b})^M \cap X$ is large in $X\}$ is A-definable.*

Proof. By Lemma 1.6. □

Remark 1.14. Note that Proposition 1.13 is a kind of definability of types for o-minimal structures. For

$$\begin{aligned} &\{\bar{b}: \phi(\bar{x}, \bar{b}) \cap X \text{ is large in } X\} \\ &\quad = \{b: \text{for } \textit{every} \text{ generic point } \bar{a} \text{ of } X \text{ over } \bar{b},\ \models \phi(\bar{a}, \bar{b})\} \\ &\quad = \{\bar{b}: \text{for } \textit{every}\ p \in S_n(M) \text{ with } X \in p \text{ and } \dim p = \dim X, \\ &\qquad\quad \phi(\bar{x}, \bar{b}) \in p\}\,. \end{aligned}$$

The only difference with ω-stable theories is that in our context a definable set X may have many generics over A up to A-isomorphism.

Note. An equivalent definition of dimension in o-minimal structures was given by van den Dries [10] who also observed many of the facts in this section.

2. Definable groups

G is here a group definable in M. Namely the universe of G is a definable subset of M^n, some n, and the group operation is also definable. We assume that G is defined over $\emptyset$.

Lemma 2.1. (i) *Let $b \in G$ and let a be a generic of G over b. Then $b \cdot a$ is a generic of G (over b).*

(ii) *For any $b \in G$ there are generics b_1, b_2 of G such that $b = b_1 \cdot b_2$.*

Proof. (i) $\dim G = \dim(a/b) = \dim(b \cdot a/b)$.

(ii) Let b_1 be generic over b. So b_1^{-1} is generic over b and by (i) so is $b_1^{-1} \cdot b$. □

We include the following lemma for interest, and the proof is left to the reader.

Lemma 2.2. *Let X be a definable subset of G. Then $\{a \in G: a \cdot X \cap X$ is large in $X\}$ is a definable subgroup of G.* □

Lemma 2.3. *Let H be a definable subgroup of G. Then* $\dim H = \dim G$ *if and only if H has finite index in G.*

Proof. By Lemma 1.5(i), dim H = dim $a \cdot H$ for any $a \in G$. So by Proposition 1.8, if dim H = dim G, then H has finite index in G. Conversely if H has finite index in G then $G = a_1 \cdot H \cup \cdots \cup a_r \cdot H$ so by Lemma 1.5(ii), dim G = dim H. □

The following crucial lemma was also proved (independently) by van den Dries (different proof).

Lemma 2.4. *Let X be a large definable subset of G. Then finitely many translates of X cover G.*

Proof. Let $M_0 \prec M$ be a small model over which X is defined. We will show

(*) For any $a \in G$ there is $b \in G^{M_0}$ such that $a \in b \cdot X$.

It will follow by compactness that there are $b_1, \ldots, b_r \in G^{M_0}$ such that every $a \in G$ is in $b_i \cdot X$ for some $i = 1, \ldots, r$.

So to prove (*), first let c be a generic point of G over M_0 such that moreover tp$(c/M_0 \cup a)$ is finitely satisfiable in M_0. It follows that c is a generic of G over $M_0 \cup a$. (For let c_1 be a maximal subtuple of c algebraically independent over M_0, and a_1 a maximal subtuple of a algebraically independent over M_0. Then easily $a_1 \frown c_1$ is algebraically independent over M_0, so $\dim(a \frown c/M_0) = \dim(a/M_0) + \dim(c/M_0)$ and so $\dim(c/M_0) = \dim(c/M_0 \cup a)$.)

Now as X is large in G it follows from Lemma 1.5(i) that $X \cdot a^{-1}$ is large in G. So by Lemma 1.13, $c \in X \cdot a^{-1}$. Thus $a \in c^{-1} \cdot X$. As tp$(c/M_0 \cup a)$ is finitely satisfiable in M_0, there is $b \in G^{M_0}$ such that $a \in b \cdot X$. This proves (*) and thus also the proposition. □

We will now show that G can be definably made into a topological group which is 'locally Euclidean'. Weil [11] showed that an algebraic group over an algebraically closed field can be recovered or defined from 'birational data', i.e., from a variety on which an algebraic group structure is given only *generically*. The second part of his proof consists in showing that from a group chunk (previously obtained), which is an open set of the variety with various desirable properties, an enveloping algebraic group can be defined. Van den Dries observed that this result can be used to show that a group definable in an algebraically closed field (characteristic 0) can be definably given geometric structure making it an algebraic group. Hrushovsky [2] gave an elegant proof of this last result of Weil's in the special case that the group chunk is a set of maximal Morley rank and degree in an *already given definable group*. It is this proof of Hrushovsky which applies almost word for word to the present context, given the machinery developed so far and results in [3].

Proposition 2.5. *Let G be a group $\emptyset$ definable in M with* dim $G = n$. *Then there are a large $\emptyset$-definable subset V of G and a topology t on G such that*

(i) *G with the topology t is a topological group (i.e. inversion and multiplication are continuous operations)*;

(ii) *V is a finite disjoint union of $\emptyset$-definable sets $U_1, \ldots, U_r$ such that for each $i = 1, \ldots, r$, U_i is t-open in G and there is a $\emptyset$-definable (in M) homeomorphism between U_i (with its t-topology) and some* open *subset V_i of M^n.*

Note. By Lemma 2.4, finitely many translates of V cover G. Thus the topology t on G is explicitly definable, and also G fits into the set-up described in Remark 1.10.

Proof. First we can by [3] write G as a finite disjoint union of $\emptyset$-definable cells. Let $U_1, \ldots, U_r$ be the cells of dimension n. Thus $V_0 = U_1 \dot\cup \cdots \dot\cup U_r$ is large in G. Each U_i is, by Fact 0.1, $\emptyset$-definably homeomorphic with an open set in M^n, and we will identify U_i with this open set. Moreover, we will identify V_0 topologically with the disjoint union of these open sets U_i. Now for each i, j we can by [3] write U_i as a finite disjoint union of $\emptyset$-definable cells on each of which either inversion is not a map into U_j, or inversion is a continuous map into U_j. Noting that

(a) For each generic a of G over $\emptyset$ there are i, j with $a \in U_i$, $a^{-1} \in U_j$ (by Lemma 1.12), and

(b) Every $\emptyset$-definable subset (of G) of dimension n contains a generic of G over $\emptyset$,

we see that for each i there are open subsets of U_i, $U_i^1, \ldots, U_i^r$ such that $\bigcup_{j=1}^r U_i^j$ is large in U_i and inversion is a continuous map from $U_i^j \to U_j$ for every j.

Put $V_1 = \dot\bigcup_{i,j} U_i^j$ (with induced topology), then summing up

(i) V_1 is large and open in V_0 and inversion is a continuous map $V_1 \to V_0$. Similarly by [3] we can find an open large $\emptyset$-definable $Y_0 \subset V_0 \times V_0$ such that

(ii) Multiplication is a continuous map from $Y_0 \to V_0$.

(As above Y_0 is obtained as a disjoint union of open sets $Y_{i,j}^k \subset U_i \times U_j$ such that multiplication is continuous from $Y_{i,j}^k \to U_k$. We use the fact that for mutually generic a, b of G, (a, b) is a generic of $G \times G$ and $a \cdot b$ is a generic of G, and use again Lemma 1.12). *Note also that Y_0 is large in $G \times G$.*

Now define $V_1' = \{a \in V_1$: for every generic b of G over a, $(b, a) \in Y_0$ and $(b^{-1}, b \cdot a) \in Y_0\}$.

By Proposition 1.13 (and Remark 1.14), V_1' is $\emptyset$-definable. We claim, in fact, that V_1' is large in G. For this, it is enough, by Lemma 1.12 to show that V_1' contains every generic a of G over $\emptyset$. But if a is a generic of G, then $a \in V_1$ (as V_1 is large and $\emptyset$-definable), and moreover for b generic of G over a, it is easy to see that both (b, a) and $(b^{-1}, b \cdot a)$ are generics of $G \times G$, so in Y_0.

Again, by partitioning V_1' into $\emptyset$-definable subcells of the U_i and throwing away cells of dimension less than n, we obtain $\emptyset$-definable $V_2 \subset V_1'$ which is open in V_0 and large in G. Now by (i) V_2^{-1} is also open in V_0, and clearly large in G.

Put $V = V_2 \cap V_2^{-1}$. Then V is open in V_0, large in G and $V = V^{-1}$.

Thus, also $V \times V$ is open in $V_0 \times V_0$ and large in $G \times G$.

Let $Y = (V \times V) \cap \{(a, b) \in Y_0 : a \cdot b \in V\}$. By (ii) above and the openness of V in V_0, Y is open in $V_0 \times V_0$ and moreover, as for mutually generic a, b of G, $(a, b) \in Y_0$ and $a \cdot b$ is generic, so in V, we see that Y is large in $G \times G$.

So without loss of generality, we have:

(iii) $V = U_1 \dot\cup \cdots \dot\cup U_r$ where the U_i are open $\emptyset$-definable subsets of M^n, and V has the induced topology.

(iv) V is large in G.

(v) Y is large in $G \times G$ (and $\emptyset$-definable).

(vi) Inversion is a continuous map from V onto V.

(vii) Y is dense open in $V \times V$ and multiplication is a continuous map from Y to V.

(viii) For any $a \in V$, if b is a generic of V over a, then $(b, a) \in Y$ and $(b^{-1}, b \cdot a) \in Y$.

((vi), (vii), (viii) are the group chunk axioms). At this point we copy Hrushovsky and so we are brief.

Lemma. (a) *For any $a, b \in G$, the set $Z = \{x \in V : a \cdot x \cdot b \in V\}$ is open in V, and the map $x \to a \cdot x \cdot b$ is a homeomorphism (in V) $Z \to a \cdot Z \cdot b$.*

(b) *For any $a, b \in G$, the set $Z = \{(x, y) \in V \times V : a \cdot x \cdot b \cdot y \in V\}$ is open in $V \times V$ and the corresponding map $Z \to V$ is continuous.*

Proof. (a) Let $x_0 \in Z$. We show that x_0 is in some open set Z_0 of V with $Z_0 \subset Z$, and that moreover the map $x \to a \cdot x \cdot b$ is continuous on Z_0. First write $b = b_1 \cdot b_2$ with $b_1, b_2 \in V$ (by Lemma 2.1, as every generic is in V). Let $c \in G$, with c generic over $\{a, x_0, b_1, b_2\}$. So $c \in V$ and $c \cdot a \in V$.

$$\text{Let } Z_0 = \{x \in V : (c \cdot a, x) \in Y, (c \cdot a \cdot x, b_1) \in Y, (c \cdot a \cdot x \cdot b_1, b_2) \in Y \text{ and } (c^{-1}, c \cdot a \cdot x \cdot b_1 \cdot b_2) \in Y\} .$$

Then by (vii), Z_0 is open in V, $Z_0 \subset Z$ and multiplication $x \to a \cdot x \cdot b (= c^{-1} \cdot c \cdot a \cdot x \cdot b_1 \cdot b_2)$ is continuous from $Z_0 \to V$. By (viii) moreover $x_0 \in Z_0$. This is sufficient. (b) is proved likewise. □

Now, define the topology t on G by: $Z \subset G$ is t-open iff for all $g \in G$, $g \cdot Z \cap V$ is open. Clearly this topology is explicitly definable (in M).

To finish the proof of Proposition 2.5,

Claim I. *Let $Z \subset V$ and $a \in G$. Then $a \cdot Z$ is t-open iff Z is open in V. (So in particular Z is t-open iff Z is open).*

Proof. If $a \cdot Z$ is t-open, then $a^{-1} \cdot (a \cdot Z) \cap V = Z$ is open in V. Conversely, if Z is open in V, then for any $g \in G$, $g \cdot (a \cdot Z) \cap V = (g \cdot a) \cdot Z \cap V$ is open in V by part (a) of the above lemma, so $a \cdot Z$ is t-open. □

Claim II. *Inversion is a t-homeomorphism on G.*

Proof. Let W be t-open in G. As finitely many translates of V cover G, we may assume (by Claim I) that $W \subset a \cdot V$ for some $a \in G$. By Claim I, $a^{-1} \cdot W$ is open in V, and so by (vi) $(a^{-1} \cdot W)^{-1} = W^{-1} \cdot a$ is open in V. Then for any $g \in G$, $g \cdot W^{-1} \cap V = g \cdot (W^{-1} \cdot a)a^{-1} \cap V$ is open in V by part (a) of the lemma. So W^{-1} is t-open. □

Claim III. *Multiplication is t-continuous on G.*

Proof. Let $W \subset G$ be t-open. We must show that $\{(x, y) \in G \times G: x \cdot y \in W\}$ is t-open in $G \times G$. As finitely many translates of V cover G, we may assume that $W \subset c \cdot V$ for some $c \in G$ and just show that for any $a, b \in G$, $Z = \{(x, y) \in a \cdot V$: $x \cdot y \in W\}$ is t-open. But $Z = \{a \cdot w, b \cdot z) \in G \times G: (w, z) \in V \times W$ and $c^{-1} \cdot a \cdot x \cdot b \cdot y \in c^{-1} \cdot W\}$. By Claim I and part (b) of the lemma, Z is t-open. □

Claims I, II and III, together with (iii) above complete the proof of Proposition 2.5. □

Remark 2.6. It is easy to see that Proposition 2.5 is valid for any o-minimal M (not only saturated M). In particular, if the order type of M is $(\mathbb{R}, <)$, then G with the topology t is a locally Euclidean topological group, and thus by Montgomery, Zippin and Gleason's solution to Hilbert's 5th problem, a Lie group.

In the special case where $M = (\mathbb{R}, <, +, \cdot)$, it is known that if U is an open definable subset of M^n and $f: U \to M^k$ is definable, then there is an open dense definable subset U^1 of U on which f is *analytic*. Thus in this case, we could in the proof of Proposition 2.5 choose V and Y such that multiplication $Y \to V$, and inversion $V \to V$ are *analytic* functions. The proof of Proposition 2.5 then definably presents G as an analytic group. That is, a semialgebraic group (i.e. a group definable in the field of real numbers) can be semialgebraically equipped with analytic structure making G a Lie group.

We now use Proposition 2.5 to show that certain 'Morley degree' style arguments go through for G. G remains a group definable in M. We let t be the topology on G given by Proposition 2.5 and V the large subset of G also given by Proposition 2.5. In the following, by *definable* we will mean definable in M, although we will make subsequent comments about what happens when we restrict ourselves to sets definable only in the structure $(G, \cdot)$.

Lemma 2.7. *Any definable set $Z \subset G$ is a finite union of t-locally closed definable subsets of G.*

Proof. First, any definable subset of V is a finite union of definable cells, each of which is locally closed (i.e. the intersection of an open set and a closed set) in M^k. As finitely many translates of V cover G, we obtain the lemma. □

Corollary 2.8. *Any definable subgroup H of G is t-closed.*

Proof. Easy, as in [7, Proposition 2.7]. □

Lemma 2.9. *Any definable subset X of G is a finite disjoint union of definably t-connected definable sets.*

Proof. As finitely many translates of V cover G, we can write X as a disjoint union $X_1 \dot\cup \cdots \dot\cup X_k$ where each $X_i \subset a_i \cdot V$ for some a_i. Translating back to V and using Fact 0.1(i) and (ii) gives us the conclusion. □

Corollary 2.10. *For any definable subset X of G and $a \in X$ there is a unique maximal definably t-connected definable subset of X containing a, which we call the definably t-connected component of a in X. The definably t-connected components of elements of X in X form a* finite *partition of X.*

Proof. An easy consequence of Lemma 2.9. □

Lemma 2.11. *Let $H \subset K$ be definable subgroups of G. Then the following are equivalent*:

(i) *H is t-open in K;*

(ii) *H has finite index in K;*

(iii) $\dim H = \dim K$.

Proof. (ii) and (iii) were proved equivalent in Lemma 2.3.

If H is t-open in K, then H is t-clopen in K, so by Corollary 2.10, H has finite index in K. Conversely, if H has finite index in K, then by Lemma 2.7 and [7, Fact 2.2], some coset of H in K has t-interior in K, and thus H is t-open in K. □

Proposition 2.12. *Let H be a definable subgroup of G. Let X be the definably t-connected component of the identity in H. Then X is the smallest definable subgroup of H of finite index in H.*

Proof. By Corollary 2.10, $X = aX$ for any $a \in H$. Thus X is a subgroup. Again by Corollary 2.10, X has finite index in G, and by Lemma 2.11, as X is definably connected, X has no proper definable subgroup of finite index. □

Remark 2.13. (i) The above results 2.8–2.12 remain valid, if we interpret 'definable' as definable in $(G, \cdot)$ and 'definably t-connected' by having no proper t-clopen definable (in $(G, \cdot)$) subset.

(ii) By Lemma 2.11 and Proposition 2.12, G has the DCC on definable subgroups.

To show that all our results from [7] apply to G, we point out

Lemma 2.14. *Let X be a definable subset of G. Then* $\dim X \geq k+1$ *iff there is definable $Y \subset X$ such that* $\dim Y \geq k$ *and Y has no t-interior in X.*

Proof. By Proposition 1.9 and the fact that G is covered by finitely many translates of V. □

So, as G with the topology t satisfies Lemmas 2.7, 2.9 and 2.14, G is 'topologically totally transcendental' (in the language of [7]) and from [7] we conclude

Corollary 2.15. (i) *G has an infinite definable abelian subgroup.*

(ii) *If* $\dim G = 1$, *then G is abelian-by-finite.* □

Corollary 2.16. *An o-minimal group (i.e. an o-minimal structure $(G, <, \cdot)$ such that $\cdot$ is a group operation) is abelian-by-finite.* □

We expect that Cherlin's description [1] of nonnilpotent groups of Morley rank 2 will go through for G with $\dim G = 2$ (and with the field K real closed instead of algebraically closed). However, at dimension 3, the similarity breaks down, as Ali Nesin's analysis of $SO_3(\mathbb{R})$ [5] has shown.

3. Definable fields

Proposition 3.1. *Let $(K, +, \cdot)$ be a field definable in M with* $\dim(K) = n$. *Then there are a large definable subset V of K and a topology t on K such that V is t-open in K is definable to an open subset, and such that with respect to t, K is a topological field (addition and multiplication are t-continuous $K \times K \to K$, additive inversion is t-continuous $K \to K$, and multiplicative inversion is t-continuous $K^* \to K^*$), and V satisfies Proposition* 2.5(ii) *(for K in place of G). (K^* denotes K-$\{0\}$).*

Proof. We start by modifying the argument in the first part of the proof of Proposition 2.5 to obtain $\emptyset$-definable $V \subset K$, and $\emptyset$-definable $Y \subset V \times V$ satisfying

(iii) $V = U_1 \dot{\cup} \cdots \dot{\cup} U_r$ where U_i are open $\emptyset$-definable subsets of M^n and V has the induced topology.

(iv) V is large in K.

(v) Y is large in $K \times K$.

(vi) Both additive and multiplicative inversion are continuous maps from V onto V.

(vii) Y is dense open in $V \times V$ and both addition and multiplication are continuous maps $Y \to V$.

(viii) If $a \in V$ and b is a generic of V over a, then $(b, a) \in Y$, $(b, b+a) \in Y$ and $(b^{-1}, b \cdot a) \in Y$.

V is easily obtained, by first choosing V_0 as in the proof of Proposition 2.5, using o-minimality to find open large $Y_0 \subset V_0 \times V_0$ such that both addition and multiplication are continuous $Y_0 \to V_0$, and finding open large $V_1 \subset V_0$ such that both additive and multiplicative inversion are continuous $V_1 \to V_0$. Define

$$V_1' = \{a \in V_1\colon \text{for every generic } b \text{ of } K \text{ over } a,\ (b, a) \in Y_0,$$
$$(-b, b+a) \in Y_0,\ (b^{-1}, b \cdot a) \in Y_0\}\ .$$

Choose V_2 as in the proof of Proposition 2.5, and then put

$$V = V_2 \cap -V_2 \cap (V_2)^{-1} \cap -(V_2)^{-1}\ .$$

Define $Y = (V \times V) \cap \{(a, b) \in Y_0\colon a + b \in V \text{ and } a \cdot b \in V\}$. Then (iii)–(viii) are satisfied. Define the topology t on K by additive translates of V, i.e. $X \subset K$ is t-open iff for all $k \in K, (k + X) \cap V$ is open in V. Then by the proof of Proposition 2.5 and (iii)–(viii) above $(K, +)$ is a topological group with respect to t, and the t-topology on V agrees with the original topology on V. To continue we need:

Lemma. (a) *Let $a \in K$. Then $X = \{x \in V\colon a \cdot x \in V\}$ is open in V and the map $x \to a \cdot x$ is continuous $X \to V$. If $a \neq 0$, then this map is a homeomorphism $X \to a \cdot X$.*

(b) *Let $a, b, c \in K$. Then $Z = \{(x, y) \in V \times V\colon c + a \cdot x + b \cdot y + x \cdot y \in V\}$ is open in V and the map $(x, y) \to c + a \cdot x + b \cdot y + x \cdot y$ is continuous from $Z \to V$.*

Proof. (a) is proved as in Proposition 2.5.

(b) Let $(x_0, y_0) \in Z$. Let $d \in K$ be generic over $\{a, b, c, x_0, y_0\}$. Let

$$Z_0 = \{(x, y) \in V \times V\colon (d \cdot a, x) \in Y, (d \cdot b, y) \in Y, (d, x) \in Y\ ,$$
$$(d \cdot x, y) \in Y, d \cdot c + d \cdot a \cdot x + d \cdot b \cdot y + d \cdot x \cdot y \in V, \text{ and}$$
$$(d^{-1}, d \cdot c + d \cdot a \cdot x + d \cdot b \cdot y + d \cdot x \cdot y) \in Y\}\ .$$

Then $Z_0 \subset Z$ (easily), Z_0 is open in V (by (vii), (viii), part (a) and the fact that addition is t-continuous).

The map $(x, y) \to c + a \cdot x + b \cdot y + x \cdot y$ is continuous $Z_0 \to V$. Moreover

$(x_0, y_0) \in Z_0$ (by (viii) and the fact that d is generic over anything $\emptyset$-definable from a, b, c, x_0, y_0). So this proves part (b) of the lemma. □

To finish:

Claim I. *Multiplication is t-continuous from $K \times K \to K$.*

Proof. Again, as finitely many additive translates of V cover K it is enough to show that for t-open $W \subset c + V$ and $a, b \in K$, $\{(x, y) \in (a + V) \times (b + V): x \cdot y \in W\}$ is t-open. This amounts to showing that

$$\{(w, z) \in V \times V: (a + w) \cdot (b + z) \in W\}$$
$$= \{(w, z) \in V \times V: (-c + a \cdot b) + a \cdot z + b \cdot w + w \cdot z \in -c + W\}$$

is open in V, which is given by (b) of the above lemma. □

Claim II. *Multiplicative inversion is a t-homeomorphism $K^* \to K^*$.*

Proof. Like Claim II in the proof of Proposition 2.5, using here the fact that multiplicative translation is continuous (Claim I above) and that finitely many multiplicative translates of V cover K^* (Lemma 2.4). □

So Proposition 3.1 is proved. □

We now work towards showing that K is real closed or algebraically closed.

Lemma 3.2. *Let G be a group definable in M. Let $f: G \to G$ be a definable endomorphism of G with finite kernel. Then* Im f *has finite index in G.*

Proof. Let a be a generic of G over the parameters A needed to define f. Then a is algebraic over $\{f(a)\} \cup A$, so $\dim(a/A) = \dim(f(a)/A)$. So $\dim(\mathrm{Im}\, f) = \dim G$, and so by Lemma 2.3, Im f has finite index in G. □

Let now K be an infinite definable field in M, equipped with the topology t given by Proposition 3.1.

Lemma 3.3. *K has no proper definable additive subgroup of finite index.*

Proof. As for stable fields, using the fact (Proposition 2.12) that K has a smallest definable additive subgroup of finite index, (namely if H has finite index in K, then $\bigcap\{k \cdot H: k \in K^*\}$ is a definable ideal of K of finite index, so equals K). □

Corollary 3.4. *K is definably t-connected.*

Proof. By Proposition 2.12. □

Lemma 3.5. *Let* $\dim K = n \geq 2$. *Then K^* is definably t-connected.*

Proof. If not, then there is a definable $X \subset K^*$, t-clopen in K^*. Now 0 must be a boundary point of X in K, otherwise X is t-clopen in K, contradicting Corollary 3.4. So we may identify (definably) some t-neighborhood of 0 in K with an open box in M^n, say B. But then $X \cap B$ is a proper nonempty definable clopen subset of $B - \{0\}$. It is easy to check that if $n \geq 2$, then any open box in M^n with a point missing is definably connected. So we have a contradiction, proving the lemma. □

Corollary 3.6. *If* $\dim K \geq 2$, *then K^* has no proper definable multiplicative subgroup of finite index.*

Proof. By Lemma 3.5 and Proposition 2.12. □

Lemma 3.7. *Let the infinite field K be definable in M. Let K' be a field which is a proper finite extension of K. Then K' can also be defined in M and moreover* $\dim K' > \dim K$.

Proof. The field structure of K' can easily be defined (from K) on K^r where r = vector space dimension of K' over K, and it is easy to see that the dimension of K^r in $M = r \cdot \dim K$ (and $r \geq 2$). □

Corollary 3.8. *If* $\dim K \geq 2$, *then K is algebraically closed.*

Proof. By Lemmas 3.2, 3.3, 3.6, 3.7 and Macintyre's argument [4]. Namely by the above lemmas, for any n, the map $x \to x^n$ is onto and if char $K = p$, the map $x \to x^p - x$ is onto. Moreover the same is true for any finite extension of K. By Galois theory K is algebraically closed. □

We can now prove

Theorem 3.9. *If the infinite field K is definable in an o-minimal structure M, then K is real closed or algebraically closed.*

Proof. If K is not algebraically closed, then K has a finite extension K'. By Lemma 3.7, K' is definable in M and $\dim K' > \dim K \geq 1$. By Corollary 3.8, K' is algebraically closed. But by Artin and Schreier, any field with an algebraically closed finite extension is real closed. So K is real closed. □

Remark 3.10. In the special case when M has order type $(\mathbb{R}, <)$, then Proposition 3.1 and Corollary 3.4 show that K with the topology t is a *connected, locally compact* topological field. By Pontrjagin [9], K is homeomorphic to $\mathbb{R}$ or to $\mathbb{C}$ with their natural topologies. Thus, by the way the topology was defined on K and the invariance of dimension, we see that $\dim K = 1$ iff $K = \mathbb{R}$, and $\dim K = 2$ iff $K = \mathbb{C}$.

As a weak version of this for arbitrary M, we have

Proposition 3.11. *Let the field K be definable in o-minimal M. Then K is real closed iff* $\dim K = 1$, *and K is algebraically closed iff* $\dim K \geq 2$.

Proof. By Corollary 3.8 and Theorem 3.9 all we need to show is that if $\dim K = 1$, then K is not algebraically closed.

So let $\dim K = 1$, and equip K with its topology t from Proposition 3.1. Now note that every point a of K has a t-neighborhood which is definably homeomorphic with an open interval in M.

Lemma A. Char $K \neq 2$.

Proof. Suppose otherwise. Let a be a generic of K over $\emptyset$. So $a + a = 0$. Let I be a neighborhood of 0 (we assume I is an interval in M) and J a neighborhood of a (similarly) such that addition takes $J \times J \to I$. Now for $x \in J$ the function 1_x, where $1_x(y) = x + y$ is a homeomorphism from J into a subinterval of I, so is order preserving or reversing. In particular 1_a is order preserving or reversing. Suppose 1_a is order preserving, so, as a is generic over $\emptyset$, there is a subinterval J' of J containing a such that for all $x \in J'$, 1_x is order preserving $J \to I$. Now let $x, y \in J'$, $y > x$. So $0 = x + x = 1_x(x) < 1_x(y) = x + y = y + x = 1_y(x) < 1_y(y) = y + y$.

So $y + y \in I$, $y + y > 0$, contradicting char $K = 2$. Similarly the assumption that 1_a is order reversing leads to a contradiction. □

Lemma B. *Not every element of K is a square.*

Proof. Again, work in a neighborhood I of 0 which we assume to be an *interval* in M. Let J be an open subinterval of I of the form $(0, a)$, $a \in I$, such that $-J \subset I$. So $-J$ is an interval with left or right endpoint 0. If $-J$ is of the form $(0, b)$, then we may assume, by o-minimality and Lemma A, that for all $x \in J$, $x < -x$, or for all $x \in J$, $x > -x$.

Suppose the first holds. Choose $x \in J$ small enough that $-x \in J$, then $x < -x < --x$, contradiction. Similarly if the second holds.
So we have established that $-J$ *is of the form* $(b, 0)$, $b \in I$.

Now the function $x \to x^2$ is continuous on K. So by o-minimality there is a sub-interval $J_1 = (0, c)$ of J such that $(J_1)^2$ is a subinterval of I with one endpoint 0. Suppose first that $(J_1)^2 = (0, d)$. (So J_1 consists of 'positive' elements.) Then $-J_1 = (e, 0)$, and $(-J_1)^2 = (J_1)^2 = (0, d)$. So no square of an element in the neighborhood (e, c) of 0 is in the interval $(e, 0)$. If all elements of $(e, 0)$ were squares, then by continuity of the squaring function, and o-minimality $\sqrt{(e, 0)}$ would contain some interval with boundary point 0, contradicting the above. Similarly, if $(J_1)^2 = (d, 0)$. □

So by Lemma B, K is not algebraically closed, completing the proof of Proposition 3.11. □

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