Definability in the infinitesimal subgroup of a simple compact Lie group

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Padova, 2017-09-14

$\mathrm{SO}_3(\mathbb{R})$

Theorem (Nesin-Pillay 1991)

- *$X \subseteq \mathrm{SO}_3(\mathbb{R})^n$ is definable in the pure group $(\mathrm{SO}_3(\mathbb{R}); *)$ iff it is definable in the field $(\mathbb{R}; +, \cdot)$.*
- *More generally, same for any simple centreless compact linear algebraic group $G \leq \mathrm{GL}_n(\mathbb{R})$.*

Example

$\{(A, B) \in \mathrm{SO}_3(\mathbb{R}) : \det(A - B) > 0\}$ is definable in $(\mathrm{SO}_3(\mathbb{R}); *)$.

Sketch of proof:

- Define a copy of $\mathrm{SO}_3(\mathbb{R})$ in $(G; *)$;
- Reconstruct the field from the projective plane of involutions of $\mathrm{SO}_3(\mathbb{R})$.
- See that this yields a bi-interpretation of $(G; *)$ with $(\mathbb{R}; +, \cdot)$.

SO_3^{00}

$$(\mathcal{R};+,\cdot) := (\mathbb{R};+,\cdot)^{\mathcal{U}} \succeq (\mathbb{R};+,\cdot)$$

$$0 \to \mathfrak{m} \to \mathcal{O} \xrightarrow{\mathrm{st}} \mathbb{R} \to 0$$

$$1 \to \mathrm{SO}_3^{00} \to \mathrm{SO}_3(\mathcal{R}) \xrightarrow{\mathrm{st}} \mathrm{SO}_3(\mathbb{R}) \to 1$$

$$\mathrm{SO}_3^{00} = \mathrm{SO}_3(\mathcal{R}) \cap \begin{pmatrix} 1+\mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & 1+\mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & 1+\mathfrak{m} \end{pmatrix}$$

$(\mathrm{SO}_3^{00};*)$ is interpretable in $(\mathcal{R};+,\cdot,\mathcal{O}) \models$ *RCVF*.

Problem

Which $(\mathcal{R};+,\cdot,\mathcal{O})$*-definable subsets of* $(\mathrm{SO}_3^{00})^n$ *are* $(\mathrm{SO}_3^{00};*)$*-definable?*
Same question for G^{00} *for* $G \subseteq \mathrm{GL}_n(\mathbb{R})$ *compact?*

$(\mathrm{SO}_3^{00};*)$

- Lie algebra $\mathfrak{g}(\mathcal{R}) = \mathrm{so}_3(\mathcal{R}) = \left\{ \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \right\} \cong \mathcal{R}^3 = \{(x,y,z)\}$.
- Infinitesimal Lie algebra: $\mathfrak{g}_{\mathfrak{m}} := \mathrm{st}^{-1}(0) \cong \mathfrak{m}^3 \leq \mathcal{R}^3$.
- Matrix exponentiation yields a homeomorphism $\exp_{\mathfrak{m}} : \mathfrak{g}_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{SO}_3^{00}$.
- $\exp_{\mathfrak{m}}(X) * \exp_{\mathfrak{m}}(Y) = \exp_{\mathfrak{m}}(X+Y) + \epsilon$ where $v(\|\epsilon\|) \geq v(\|X\|) + v(\|Y\|)$.
- If X and Y are collinear then $\exp_{\mathfrak{m}}(X) * \exp_{\mathfrak{m}}(Y) = \exp_{\mathfrak{m}}(X+Y)$.
- For $x \in \mathrm{SO}_3^{00}$ and $h \in \mathrm{SO}_3(\mathcal{R})$, group conjugation $x \mapsto x^h := h * x * h^{-1}$ agrees with the matrix action of $\mathrm{SO}_3(\mathcal{R})$ on $\mathfrak{m}^3$:

$$\exp_{\mathfrak{m}}(X)^h = \exp_{\mathfrak{m}}(hX).$$

Main theorem

Theorem

$G \subseteq \mathrm{GL}_n(\mathbb{R})$ a simple compact linear algebraic group.

(i) *$X \subseteq (G^{00})^n$ is $(G^{00}; *)$-definable iff it is $(\mathcal{R}; +, \cdot, \mathcal{O})$-definable.*

(ii) *Moreover, the interpretation of $(G^{00}; *)$ in $(\mathcal{R}; +, \cdot, \mathcal{O})$ can be completed to a bi-interpretation.*

Example

$\{(A, B) \in \mathrm{SO}_3^{00} : v(\det(A - B)) > \alpha\}$ is definable in $(\mathrm{SO}_3^{00}; *)$.

Outline of proof:

(I) Find an $(\mathrm{SO}_3^{00}; *)$-definable ordered interval J;

(II) apply o-minimal trichotomy to get a field K in J;

(III) Find a copy of SO_3^{00} in G^{00};

(IV) use adjoint representation to see the pair $G^{00} \leq G(\mathcal{R})$ in K, yielding a bi-interpretation.

(I): Finding an ordered interval

- Let $S := \mathrm{SO}_3(\mathcal{R})$ and $S^{00} := \mathrm{SO}_3^{00}$.
- Let $b \in S^{00} \setminus \{e\}$.
- $C_S(b) := \{h \in S : b^h = b\} \cong \mathrm{SO}_2(\mathcal{R})$;
- $C_{S^{00}}(b) := C_S(b) \cap S^{00} \cong \mathrm{SO}_2^{00} \cong \mathfrak{m}$.
- $b^S b^S = \xi(S^2)$ where $\xi(h, h') = b^h * b^{h'}$.
- $b^S b^S = \exp_{\mathfrak{m}}(B)$ where $B \subseteq \mathfrak{m}^3$ is the closed ball of radius $\|b^2\|$.
- $b^S b^S \cap C_{S^{00}}(b)$ is the interval $[b^{-2}, b^2]$.
- By definable choice for the $(\mathcal{R}; +, \cdot)$-definable map ξ, $X := b^{S^{00}} b^{S^{00}} \cap C_{S^{00}}(b)$ contains some interval $(h, b^2]$.
- Translating, get $(S^{00}; *)$-definable interval $[e, p) \subseteq C_{S^{00}}(b)$, hence $J := (p^{-1}, p)$ as an ordered interval.
- Explicitly: $p := b^2 h^{-1}$, then $(e, p) = h^{-1} X \cap b^2 X^{-1}$.

(II): Trichotomy

- $T_e(J^{S^{00}})$ spans $\mathcal{R}^3$, so for appropriate $h_1, h_2 \in S^{00}$ and after shrinking J,

$$\phi : J^3 \to S^{00}; \phi(x_0, x_1, x_2) = x_0 * x_1^{h_1} * x_2^{h_2}$$

  is a bijection with a neighbourhood of $e \in S^{00}$.
- $(J; *, <)$ and ϕ are definable both in $(S^{00}; *)$ and in $(\mathcal{R}; +, \cdot)$.
- Pulling back the S^{00} group structure via ϕ puts "non-linear" structure on J at e.
- By the Peterzil-Starchenko o-minimal trichotomy, a real closed field $(K; +, \cdot)$ on an interval $e \in K \subseteq J$ is definable in this structure on J.
- So $(K; +, \cdot)$ is definable both in $(S^{00}; *)$ and in $(\mathcal{R}; +, \cdot)$.

(III): Finding an SO_3^{00} in G^{00}

- $\mathfrak{g}_0 := L(G)$
- $\mathfrak{h}_0 \leq g_0$ Cartan subalgebra (i.e. maximal abelian).
- $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{h} := \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$.
- $X \in \mathfrak{g}$ an $\mathrm{ad}_{\mathfrak{h}}$-eigenvector for a root $\alpha \in \mathfrak{h}^* \setminus \{0\}$; i.e. $[H, X] = \alpha(H)X$ for $H \in \mathfrak{h}$.
- Set $U := X - \bar{X}$, $V := iX + i\bar{X}$.
- Since G is compact, $U, V \in \mathfrak{g}_0$, and $[U, V] \in \mathfrak{h}_0$, and $\mathfrak{s} := \langle U, V, [U, V] \rangle \cong \mathrm{so}_3$. Let $\mathfrak{s}' := \mathfrak{h}_0 + \mathfrak{s}$.
- Let $S, S' \leq G$ with $L(S) = \mathfrak{s}, L(S') = \mathfrak{s}'$.
- So $S \cong \mathrm{SO}(3)$ or $S \cong \mathrm{Spin}(3)$ and $S^{00} \cong \mathrm{SO}_3^{00}$.
- Considering root space decomposition, calculate: $\mathfrak{s}' = C_{g_0}(C_{g_0}(\mathfrak{s}'))$, and $\mathfrak{s} = [\mathfrak{s}', \mathfrak{s}']$. Deduce: $S' = C_G(C_G(S'))$ and $S = (S', S')_1$; $S'^{00} = C_{G^{00}}(C_{G^{00}}(S'^{00}))$ and $S^{00} = (S'^{00}, S'^{00})_1$.
- So S is $(\mathcal{R}; +, \cdot)$-definable, S^{00} is $(G^{00}; *)$-definable.
- (Trichotomy argument to find K works when $S = \mathrm{Spin}(3)$.)

(IV): Bi-interpretation

- $(K; +, \cdot)$ is definable both in $(G^{00}; *)$ and in $(\mathcal{R}; +, \cdot)$.
- Otero-Peterzil-Pillay: exists $(\mathcal{R}; +, \cdot)$-definable isomorphism $\theta : (\mathcal{R}; +, \cdot) \xrightarrow{\sim} (K; +, \cdot)$.
- θ induces $\theta_G : G(\mathcal{R}) \xrightarrow{\sim} G(K)$.

Claim

$\theta_G \restriction_{G^{00}} : G(\mathcal{R})^{00} \xrightarrow{\sim} G(K)^{00}$ is $(G^{00}; *)$-definable.

Proof of Main Theorem.

- $\mathcal{O}$ is definable in $(\mathcal{R}; +, \cdot, G^{00})$,
- so $(\mathcal{R}; +, \cdot, \mathcal{O})$ is interpreted on K in $(G^{00}; *)$ via θ, since $G(K)^{00}$ is $(G^{00}; *)$-definable by the claim.
- $(G^{00}; *)$ is interpreted in $(\mathcal{R}; +, \cdot, \mathcal{O})$ tautologically.
- The composed interpretations are θ and $\theta_G \restriction_{G^{00}}$, which are definable in $(\mathcal{R}; +, \cdot, G^{00})$ resp. $(G^{00}; *)$.

□

Proof of claim

Claim

$\theta_G\restriction_{G^{00}}: G(\mathcal{R})^{00} \xrightarrow{\sim} G(K)^{00}$ is $(G^{00};*)$-definable.

Proof.

- Quotienting by the discrete centre, we may assume G is centreless. Let ϕ be a chart for G in J as above. (Exists by simplicity of G.)
- Differentiation in K yields via ϕ an adjoint embedding $\mathrm{Ad} : G(\mathcal{R}) \to \mathrm{GL}_d(K)$.
- Ad is $(\mathcal{R};+,\cdot)$-definable.
- $\mathrm{Ad}\restriction_{G^{00}}$ is $(G^{00};*)$-definable.
- $\eta := \mathrm{Ad} \circ \theta_G^{-1} : G(K) \to \mathrm{GL}_d(K)$ is $(K;+,\cdot)$-definable by purity, hence $(G^{00};*)$-definable.
- So $\theta_G\restriction_{G^{00}} = \eta^{-1} \circ \mathrm{Ad}\restriction_{G^{00}}$ is $(G^{00};*)$-definable.

□