Definability in the group of infinitesimal rotations

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SO_3

Theorem (Nesin-Pillay 1991)

- *$X \subseteq \mathrm{SO}_3(\mathbb{R})^n$ is definable in the pure group $(\mathrm{SO}_3(\mathbb{R}); *)$ iff it is definable in the field $(\mathbb{R}; +, \cdot)$.*
- *More generally, same for any simple centerless compact Lie group G definable in $(\mathbb{R}; +, \cdot)$.*

Example

$\{(A, B) \in \mathrm{SO}_3(\mathbb{R}) : \det(A - B) > 0\}$ is definable in $(\mathrm{SO}_3(\mathbb{R}); *)$.

Sketch of proof:

- Define a copy of $\mathrm{SO}_3(\mathbb{R})$ in $(G; *)$;
- Reconstruct the field from the projective plane of involutions of $\mathrm{SO}_3(\mathbb{R})$.
- See that this yields a bi-interpretation of $(G; *)$ with $(\mathbb{R}; +, \cdot)$.

SO_3^{00}

$$(\mathcal{R}; +, \cdot) := (\mathbb{R}; +, \cdot)^{\mathcal{U}} \succeq (\mathbb{R}; +, \cdot)$$

$$0 \to \mathfrak{m} \to \mathcal{O} \xrightarrow{\text{st}} \mathbb{R} \to 0$$

$$1 \to \mathsf{SO}_3^{00} \to \mathsf{SO}_3(\mathcal{R}) \xrightarrow{\text{st}} \mathsf{SO}_3(\mathbb{R}) \to 1$$

$$\mathsf{SO}_3^{00} = \mathsf{SO}_3(\mathcal{R}) \cap \begin{pmatrix} 1+\mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & 1+\mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & 1+\mathfrak{m} \end{pmatrix}$$

$(\mathsf{SO}_3^{00}; *)$ is interpretable in $(\mathcal{R}; +, \cdot, \mathcal{O}) \models$ *RCVF*.

Problem

Which $(\mathcal{R}; +, \cdot, \mathcal{O})$-definable subsets of $(\mathsf{SO}_3^{00})^n$ are $(\mathsf{SO}_3^{00}; *)$-definable?

$(\mathrm{SO}_3^{00}; *)$

- Lie algebra $\mathfrak{g}(\mathcal{R}) = \mathrm{so}_3(\mathcal{R}) = \left\{ \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \right\} \cong \mathcal{R}^3 = \{(x, y, z)\}$.
- Infinitesimal Lie algebra: $\mathfrak{g}_\mathfrak{m} := \mathrm{st}^{-1}(0) \cong \mathfrak{m}^3 \leq \mathcal{R}^3$.
- Matrix exponentiation yields a homeomorphism $\exp_\mathfrak{m} : \mathfrak{g}_\mathfrak{m} \xrightarrow{\sim} \mathrm{SO}_3^{00}$.
- $\exp_\mathfrak{m}(X) * \exp_\mathfrak{m}(Y) = \exp_\mathfrak{m}(X + Y) + \epsilon$ where $v(\|\epsilon\|) \geq v(\|X\|) + v(\|Y\|)$.
- If X and Y are collinear then $\exp_\mathfrak{m}(X) * \exp_\mathfrak{m}(Y) = \exp_\mathfrak{m}(X + Y)$.
- For $x \in \mathrm{SO}_3^{00}$ and $h \in \mathrm{SO}_3(\mathcal{R})$, group conjugation $x \mapsto x^h := h * x * h^{-1}$ agrees with the matrix action of $\mathrm{SO}_3(\mathcal{R})$ on $\mathfrak{m}^3$:

$$\exp_\mathfrak{m}(X)^h = \exp_\mathfrak{m}(hX).$$

Main theorem

Theorem

(i) *$X \subseteq (\mathrm{SO}_3^{00})^n$ is $(\mathrm{SO}_3^{00}; *)$-definable iff it is $(\mathcal{R}; +, \cdot, \mathcal{O})$-definable.*

(ii) *Moreover, the interpretation of $(\mathrm{SO}_3^{00}; *)$ in $(\mathcal{R}; +, \cdot, \mathcal{O})$ can be completed to a bi-interpretation.*

Example

$\{(A, B) \in \mathrm{SO}_3^{00} : v(\det(A - B)) > \alpha\}$ is definable in $(\mathrm{SO}_3^{00}; *)$.

Outline of proof:

- Find an $(\mathrm{SO}_3^{00}; *)$-definable ordered interval J;
- apply trichotomy to get a field K in J;
- use adjoint representation to see the pair $\mathrm{SO}_3^{00} \leq \mathrm{SO}_3(\mathcal{R})$ in K, yielding a bi-interpretation;
- the characterisation of definable sets follows.

Finding an ordered interval

- Let $G := \mathrm{SO}_3(\mathcal{R})$ and $G^{00} := \mathrm{SO}_3^{00}$.
- Let $b \in G^{00} \setminus \{e\}$.
- $C_b^G := \{h \in G : h * b = b * h\} \cong \mathrm{SO}_2(\mathcal{R})$;
- $C_b^{G^{00}} := C_b^G \cap G^{00} \cong \mathrm{SO}_2^{00} \cong \mathfrak{m}$.
- $b^G b^G = \xi(G^2)$ where $\xi(h, h') = b^h * b^{h'}$.
- $b^G b^G = \exp_{\mathfrak{m}}(B)$ where $B \subseteq \mathfrak{m}^3$ is the closed ball of radius $\|b^2\|$.
- $b^G b^G \cap C_b^{G^{00}}$ is the interval $[b^{-2}, b^2]$.
- By definable choice for the $(\mathcal{R}; +, \cdot)$-definable map ξ, $X := b^{G^{00}} b^{G^{00}} \cap C_b^{G^{00}}$ contains some interval $[h, b^2]$.
- Translating, get $(G^{00}; *)$-definable interval $[e, p) \subseteq C_b^{G^{00}}$, hence $J := (p^{-1}, p)$ as an ordered interval.
- Explicitly: $p := b^2 h^{-1}$, then $[e, p] = h^{-1} X \cap b^2 X^{-1}$.

Trichotomy

- $b^{G^{00}}$ spans $\mathcal{R}^3$, so for appropriate $h_1, h_2 \in G^{00}$ and after shrinking J,

$$\phi : J^3 \to G^{00}; \phi(x_0, x_1, x_2) = x_0 * x_1^{h_1} * x_2^{h_2}$$

  is a bijection with a neighbourhood of $e \in G^{00}$.
- $(J; *, <)$ and ϕ are definable both in $(G^{00}; *)$ and in $(\mathcal{R}; +, \cdot)$.
- Pulling back the G^{00} group structure via ϕ puts "non-linear" structure on J.
- By the Peterzil-Starchenko o-minimal trichotomy, a real closed field $(K; +, \cdot)$ on an interval $K \subseteq J$ is definable in this structure on J.
- So $(K; +, \cdot)$ is definable both in $(G^{00}; *)$ and in $(\mathcal{R}; +, \cdot)$.

Bi-interpretation

- $(K; +, \cdot)$ is definable both in $(G^{00}; *)$ and in $(\mathcal{R}; +, \cdot)$.
- Otero-Peterzil-Pillay: exists $(\mathcal{R}; +, \cdot)$-definable isomorphism $\theta : (\mathcal{R}; +, \cdot) \xrightarrow{\sim} (K; +, \cdot)$.
- θ induces $\theta_G : G = \mathrm{SO}_3(\mathcal{R}) \xrightarrow{\sim} \mathrm{SO}_3(K)$.

Claim

$\theta_G|_{G^{00}} : \mathrm{SO}_3(\mathcal{R})^{00} \xrightarrow{\sim} \mathrm{SO}_3(K)^{00}$ is $(G^{00}; *)$-definable.

Proof of main theorem.

- $\mathcal{O}$ is definable in $(\mathcal{R}; +, \cdot, G^{00})$,
- so $(\mathcal{R}; +, \cdot, \mathcal{O})$ is interpreted on K in $(G^{00}; *)$ via θ, since $\mathrm{SO}_3(K)^{00}$ is $(G^{00}; *)$-definable by the claim.
- $(G^{00}; *)$ is interpreted in $(\mathcal{R}; +, \cdot, \mathcal{O})$ tautologically.
- The composed interpretations are θ and $\theta_G|_{G^{00}}$, which are definable in $(\mathcal{R}; +, \cdot, G^{00})$ resp. $(G^{00}; *)$.

□

Proof of claim

Claim

$\theta|_{G^{00}} : \mathrm{SO}_3(\mathcal{R})^{00} \xrightarrow{\sim} \mathrm{SO}_3(K)^{00}$ *is* $(G^{00}; *)$*-definable.*

Proof.

- Differentiation in K yields via ϕ an adjoint embedding

$$\mathrm{Ad} : \mathrm{SO}_3(\mathcal{R}) \to \mathrm{GL}_3(K)$$

- Ad is $(\mathcal{R}; +, \cdot)$-definable.
- $\mathrm{Ad}|_{G^{00}}$ is $(G^{00}; *)$-definable.
- $\eta := \mathrm{Ad} \circ \theta_G^{-1} : \mathrm{SO}_3(K) \to \mathrm{GL}_3(K)$ is $(K; +, \cdot)$-definable by purity, hence $(G^{00}; *)$-definable.
- So $\theta_G|_{G^{00}} = \eta^{-1} \circ \mathrm{Ad}|_{G^{00}}$ is $(G^{00}; *)$-definable.

□