Geometry of combinatorially extremal algebraic configurations

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1 Bounds on finite algebraic configurations

$V \subseteq \mathbb{C}^n$ irreducible algebraic set,
$N \in \mathbb{N}$,
$X_i \subseteq \mathbb{C}$ with $|X_i| \leq N$, $i = 1 \ldots n$.
Then $|V \cap \Pi_i X_i| \leq O(N^d)$ where $d = \dim(V)$.

Question 1.1. For what V is this exponent d optimal,
i.e. $|V \cap \Pi_i X_i| \not\leq O(N^{d-\epsilon})$ for any $\epsilon > 0$.

(also interested in similar questions when co-ordinates $\mathbb{C}$ are replaced by higher dimensional varieties)

Theorem 1.2 (Elekes-Szabó)**.** *For $n = 3$ and $d = 2$, the exponent 2 is optimal iff either*

- *V is in co-ordinatewise correspondence with the graph of the group operation of a 1-dimensional algebraic group G, i.e. V is a component of the Zariski closure of $\{(\alpha_1(g), \alpha_2(h), \alpha_3(g+h)) : g, h \in G\}$ where $\alpha_i : G \to \mathbb{C}$ are finite-to-finite algebraic correspondences,*
- *or V projects to a curve, i.e. $\dim(\pi_{ij}(V)) = 1$ some $i \neq j \in \{1, 2, 3\}$.*

*(Hong Wang, Raz-Sharir-deZeeuw: When 2 isn't optimal, 11/6 works.
(Could be that $1 + \epsilon$ works for any $\epsilon > 0\ldots$)
Raz-Sharir-deZeeuw: case $(n = 4, d = 3)$.)*

2 Hrushovski δ-formalism

Hrushovski "On Pseudo-Finite Dimensions" (2013)

- $K := \Pi_{i\to\mathcal{U}} K_i$,
  $\mathcal{U} \subseteq \mathcal{P}(\omega)$ non-principal ultrafilter,
  K_i expansions of $(\mathbb{C}; +, \cdot)$ in a countable language $\mathcal{L} \supseteq \mathcal{L}_{\text{ring}}$.
- $X \subseteq K^n$ is **internal** if $X = \Pi_{i\to\mathcal{U}}(X_i)$ for some $X_i \subseteq K_i^n$.

- Then $|X| := \Pi_{i \to \mathcal{U}} |X_i| \in \mathbb{R}^{\mathcal{U}} \cup \{\infty\}$.
- Let $\xi_0 \in \mathbb{R}^{\mathcal{U}}_{>0}$ with $\xi_0 > \mathbb{R}$.
- $\boldsymbol{\delta}(X) := \mathrm{st}\left(\frac{\log(|X|)}{\log(\xi_0)}\right) \in \mathbb{R}_{\geq 0} \cup \{-\infty, \infty\}$ ("coarse pseudofinite dimension").
- Note $\boldsymbol{\delta}(X \times Y) = \boldsymbol{\delta}(X) + \boldsymbol{\delta}(Y)$, and $\boldsymbol{\delta}(X \cup Y) = \max(\boldsymbol{\delta}(X), \boldsymbol{\delta}(Y))$.
- $\boldsymbol{\delta}(\phi) := \boldsymbol{\delta}(\phi(K))$.
- $\boldsymbol{\delta}(\Phi) := \inf\{\boldsymbol{\delta}(\phi) : \phi \in \Phi\}$.
- $\boldsymbol{\delta}(a/C) := \boldsymbol{\delta}(\mathrm{tp}(a/C))$.

Assume $\boldsymbol{\delta}$ is **continuous**:
Given $\phi(x, y)$ and $\alpha \in \mathbb{R}$, for $\epsilon \in \mathbb{R}_{>0}$ exists definable Y s.t.

$$\boldsymbol{\delta}(\phi(x, b)) \leq \alpha \Rightarrow b \in Y \Rightarrow \boldsymbol{\delta}(\phi(x, b)) < \alpha + \epsilon.$$

Can add quantifiers $\exists_{<\xi_0^q}$ for $q \in \mathbb{Q}$ to get continuity.

Fact 2.1.

(i) $a \equiv_C b \Rightarrow \boldsymbol{\delta}(a/C) = \boldsymbol{\delta}(b/C)$.

(ii) $\boldsymbol{\delta}(ab/C) = \boldsymbol{\delta}(a/bC) + \boldsymbol{\delta}(b/C)$.

(iii) A partial type Φ *over a countable set* C *has a realisation* $K \models \Phi(a)$ *with* $\boldsymbol{\delta}(a/C) = \boldsymbol{\delta}(\Phi)$.

Fix C_0 a countable algebraically closed subfield of K.
Assume $C_0 \subseteq \mathrm{dcl}(\emptyset)$.

Definition 2.2. For $B \subseteq K$,

- $\mathrm{acl}^0(B) := C_0(B)^{\mathrm{alg}} \leq K$;
- $\dim^0(B) := \mathrm{trd}(C_0(B)/C_0)$.

Note $a \in \mathrm{acl}^0(B) \Rightarrow \boldsymbol{\delta}(a/B) = 0$.

3 Coherent modularity

3.1 Coherence

Definition 3.1. $X \subseteq K$ is (coarsely) **coherent** if $\dim^0(\overline{a}) = \boldsymbol{\delta}(\overline{a})$ for any $\overline{a} \in X^{<\omega}$.

Remark 3.2. If $\overline{a} = a_1 \ldots a_n$ and $\boldsymbol{\delta}(a_i) = \dim^0(a_i)$,
then $\boldsymbol{\delta}(\overline{a}) \leq \dim^0(\overline{a})$,
and $\{a_1, \ldots, a_n\}$ is coherent iff $\boldsymbol{\delta}(\overline{a}) = \dim^0(\overline{a})$.

Remark 3.3. For V over $C_0 \subseteq \mathbb{C}$,
the exponent $d = \dim(V)$ is optimal iff for some such structure K exists coherent $\overline{a} \in V(K)$ with $V = \mathrm{locus}(\overline{a}/C_0)$.

Indeed: if d is optimal,
for $\epsilon > 0$, for arbitrarily large N,
have $|X_{i,N}| = N$ s.t. $|V \cap \Pi_i X_{i,N}| > N^{d-\epsilon}$.
Take K in language with $\Pi_{N \to \mathcal{U}} X_{i,N} =: X_i$ definable.
Set $\xi_0 := |X_i|$.
Then $\boldsymbol{\delta}(V \cap \Pi_i X_i) = d$,
so say $\overline{a} \in V \cap \Pi_i X_i$ with $\boldsymbol{\delta}(\overline{a}) = d$,
then $\overline{a}$ is coherent and generic in V.
Converse is similar.

3.2 Geometries

Recall a **pregeometry** is a closure operator cl on a set S satisfying exchange

$$a \in \mathrm{cl}(Cb) \setminus \mathrm{cl}(C) \Rightarrow b \in \mathrm{cl}(Ca)$$

and finite character ($\mathrm{cl}(A) = \cup_{A_0 \subseteq_{\mathrm{fin}} A} \mathrm{cl}(A_0)$).

The associated **geometry** is

$$\mathbb{P}(S) := (S \setminus \mathrm{cl}(\emptyset))/\{\mathrm{cl}(x) = \mathrm{cl}(y)\}.$$

For $A \subseteq S$, $\dim(A) = \min\{|A_0| : A_0 \subseteq A \subseteq \mathrm{cl}(A_0)\}$.

Definition 3.4. A geometry (S, cl) is **modular** if for $a, b \in S$ and $C \subseteq S$,
if $a \in \mathrm{cl}(bC) \setminus \mathrm{cl}(C)$ then exists $c \in \mathrm{cl}(C)$ such that $a \in \mathrm{cl}(bc)$.

If V is a vector space over a division ring R,
$\mathbb{P}(V) := \mathbb{P}(V; \langle\cdot\rangle_R)$, is modular.
$\mathcal{G}_K := \mathbb{P}(K; \mathrm{acl}^0)$ is not modular: $a = c_1 \cdot b + c_2$.

Definition 3.5 (SKIP).

- $(S_1, \mathrm{cl}_1), (S_2, \mathrm{cl}_2)$ geometries. The **coproduct** is the geometry $(S_1 \dot\cup S_2, \mathrm{cl}_1 \dot\cup \mathrm{cl}_2)$ where $(\mathrm{cl}_1 \dot\cup \mathrm{cl}_2)(X_1 \dot\cup X_2) = \mathrm{cl}_1(X_1) \dot\cup \mathrm{cl}_2(X_2)$ for $X_i \subseteq S_i$.
- A subgeometry of a geometry $(S; \mathrm{cl})$ is $(X; \mathrm{cl}\restriction_X)$ where $X \subseteq S$ and $\mathrm{cl}\restriction_X (A) = \mathrm{cl}(A) \cap X$.

Fact 3.6. (S, cl) *modular geometry.*

Say $a, b \in S$ *are* ***collinear*** *(equiv: non-orthogonal) if* $a \in \mathrm{cl}(bc)$ *for some* $c \neq a$.

Then (S, cl) *is the coproduct of the subgeometries on the collinearity equivalence classes,*
and each class of $\dim > 3$ *is a projective geometry* $\mathbb{P}(V)$ *over a division ring (coordinatisation theorem of projective geometry).*

3.3 Coherent modularity

Hrushovski observes that incidence bounds yield modularity.

If $y = a \cdot x + b$, $x, y, a, b \in K \setminus C_0$, then $\{x, y, a, b\}$ is not coherent:

Theorem 3.7 (Szemeredi-Trotter for $\mathbb{C}$, due to Zahl)**.** *For $P, L \subseteq \mathbb{C}^2$ with $|P|, |L| \leq N^2$,*

$$|\{((x, y), (a, b)) \in P \times L : y = a \cdot x + b\}| \leq O(N^{\frac{8}{3}}).$$

Lemma 3.8. *For $X \subseteq K$ coherent,*
$\operatorname{ccl}(X) := \{c \in \operatorname{acl}^0(X) : c$ *is coherent*$\}$ *is coherent.*

Using generalisations of Szemeredi-Trotter proved in various levels of generality by various authors (one by Elekes-Szabó suffices for our purposes),

Proposition 3.9. *If X is coherent and $X = \operatorname{ccl}(X)$,*
then $\mathcal{G}_X := \mathcal{G}(X; \operatorname{acl}^0) \subseteq \mathcal{G}_K$ is a modular geometry.

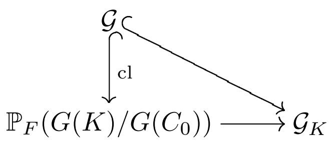
Remark 3.10. If Ba is coherent and $a \notin \operatorname{acl}^0(B)$,
then exists $a' \equiv_B a$ with $\boldsymbol{\delta}(a'/Ba) = \boldsymbol{\delta}(a'/B)$,
and then $a' \notin \operatorname{acl}^0(Ba)$,
and so Baa' is coherent.

Hence: $\mathcal{G}_X = \dot{\bigcup}_i \mathcal{G}_i$ where for each i,
$\mathcal{G}_i = \{*\}$ or $\mathcal{G}_i \subseteq \mathcal{P}(V_i) \subseteq \mathcal{G}_K$,
some V_i vector space over a division ring F_i.

3.4 Projective subgeometries of $\mathcal{G}_K$

Example 3.11. Let G be a 1-dimensional algebraic group over C_0,
let $F \leq \operatorname{End}^0_{C_0}(G) := \mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{End}_{C_0}(G)$ be a division subring.
Then $G(K)/G(C_0)$ is an F-vector space.
Let $g_1, \ldots, g_n \in G$ be independent generics.
Then $\mathbb{P}_F(\langle g_1/G(C_0), \ldots, g_n/G(C_0) \rangle_F)$ is a subgeometry of $\mathcal{G}_K$.

Theorem 3.12 (Evans-Hrushovski '91)**.** *Any projective subgeometry $\mathcal{G} \subseteq \mathcal{G}_K$ of dim ≥ 3 of the above form.*



So suppose $\overline{a}$ coherent and each (a_i, a_j) is collinear in $\operatorname{ccl}(\overline{a})$,
and $\dim(\overline{a}) > 1$.
Then there is a 1-dimensional group G and g_i acl^0-interalgebraic with a_i,
s.t. $\operatorname{locus}(\overline{g}) = \ker(M)^0$ for some $M \in \operatorname{Mat}(\operatorname{End}_{C_0}(G))$.
Same holds for $\dim(\overline{a}) = 1$, with $G := \mathcal{G}_a$ and $g_i = g_j$.
So -

Theorem 3.13. *The exponent $d = \dim(V)$ is optimal for V iff,*
up to finite-to-finite correspondences on the co-ordinates,
V is a product of algebraic subgroups of powers of 1-dimensional algebraic groups.

4 Higher dimension

Theorem 4.1 (Elekes-Szabo). *$V \subseteq W_1 \times W_2 \times W_3$, $\dim(W_i) = k$, $\dim(V) = 2k$, all irreducible complex.*
(Nice example: W_i surfaces in $\mathbb{C}^3$, V collinearity.)
*$X_i \subseteq W_i$, $|X_i| = N$, in **general position**:*
for $W_i' \not\subseteq W_i$ proper subvariety,
$|X_i \cap W_i'| \in O_{\deg(W_i')}(1)$.
Then either $|V \cap \Pi_i X_i| \leq O(N^{2-\eta})$ or V is in correspondence with a group operation (or trivial case).

Remark 4.2. Example showing necessity of general position:
$V :=$ graph of $(a_1, b_1) * (a_2, b_2) = (a_1 + a_2 + b_1^2 b_2^2, b_1 + b_2)$,
$X_{i,N} := \{-N^4, \ldots, N^4\} \times \{-N, \ldots, N\} \subseteq \mathbb{C}^2 =: W_i$.

4.1 Coarse general position

Definition 4.3. $\overline{a} \in K^{<\omega}$ is in **coarse general position** if

$$\dim^0(a/B) < \dim^0(a) \Rightarrow \boldsymbol{\delta}(a) = 0$$

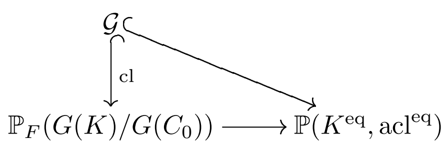
for any $B \subseteq K$.

Definition 4.4.

- $K^{\text{eq}} := \bigcup_n K^n$.
- $X \subseteq K^{\text{eq}}$ is (coarsely) **coherent** if every $a \in X$ is in coarse general position and $\dim^0(\overline{a}) = \boldsymbol{\delta}(\overline{a})$ for any $\overline{a} \in X^{<\omega}$.
- $\operatorname{ccl}(X) := \{x \in \operatorname{acl}^{\text{eq}}(X) : \{x\} \text{ is coherent}\}$.

Theorem 4.5. *Suppose $X = \operatorname{ccl}(X)$.*
Then $(X, \operatorname{acl}^{\text{eq}^0})$ is modular.

As in Evans-Hrushovski, using abelian group configuration,
$\mathcal{G}$ projective geometry of dim ≥ 4 in $\mathbb{P}(K^{\text{eq}}, \operatorname{acl}^{eq})$ factors



for some abelian algebraic group G and a division ring $F \leq \operatorname{End}^0_{C_0}(G)$.

So obtain corresponding analogue of 1-dimensional theorem, generalising Elekes-Szabo.