Quasi-homomorphisms and connected components

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A talk for a seminar in Münster, on the example in [HKP22] of an approximate subgroup X for which $\langle X\rangle^{00}$ does not exist.

1 Connected components and approximate subgroups

Work in a monster model of an $\mathcal{L}$-theory.
Definable means $\emptyset$-definable unless otherwise specified.

Let X be a definable approximate subgroup
(i.e. X is symmetric ($e \in X = X^{-1}$) and $X^2 \subseteq FX$ for some finite F).
Let $H := \langle X\rangle$ be the $\bigvee$-definable subgroup generated by X.

Examples.

(i)
- $X = [-1, 1] \subseteq (\mathbb{R}^*; +)$.
- $X + X \subseteq \{1, -1\} + X$.
- $H = \langle X\rangle \leq \mathbb{R}^*$ is the convex hull of the standard reals.

(ii)
- $X = \{-N, \ldots, N\} \subseteq \mathbb{Z}^*$ where $N \in \mathbb{N}^*$ is a non-standard integer.
- $X + X \subseteq \{-N, N\} + X$.
- $H = \langle X\rangle \leq \mathbb{Z}^*$ is the smallest convex subgroup containing N.

(iii) Let $f : G \to H$ be a quasi-homomorphism,
meaning that $F := \{f(gh)^{-1}f(g)f(h) : g, h \in G\} \subseteq H$ is finite.
Suppose also $f(g^{-1}) = f(g)^{-1}$.
(e.g. $f : \mathbb{R} \to \mathbb{Z}$ the integer part map, with $F = \{-1, 0, 1\}$.)

- $X := \Gamma_f \leq G^* \times H^*$ ($(G^*, H^*) \succ (G, H)$ monster model).
- $X^2 \subseteq (e, F) \cdot X$,
since $(g, f(g)) \cdot (h, f(h)) = (gh, f(g)f(h)) \in (e, F) \cdot (gh, f(gh))$,
and X is symmetric.

- $H = \langle X \rangle \subseteq \{(g, \langle F \rangle \cdot f(g)) : g \in G^*\}$.

$A \subseteq H$ is ($\bigwedge$-)definable iff A is a ($\bigwedge$-)definable subset of some X^n.

Definition. For H a $\bigvee$-definable group,
$H^{00} \leq H$ is the smallest $\bigwedge$-definable subgroup of bounded index, if it exists,
$H^{000} \leq H$ is the smallest invariant subgroup of bounded index if it exists.

(Bounded: cardinality bounded independently of the choice of model.)

Note that H^{00} exists iff some $\bigwedge$-definable bounded index subgroup exists.
Then H/H^{00} is a locally compact group with the logic topology,
and by Hilbert 5 one obtains a Lie group.
This was a key step ("Hrushovski's Lie model theorem") in the Breuillard-Green-Tao classification of finite approximate groups.

Fact (Massicot-Wagner)**.** *If X is definably amenable (exists an invariant finitely additive ("Keisler") measure on the definable subsets of H with $\mu(X) = 1$), then H^{00} exists and $H^{00} \leq X^8$.*

This holds in particular when X is an ultraproduct of finite K-approximate groups for a fixed K.

Remark. This follows from [MW15], which finds an H-$\bigwedge$-definable bounded index subgroup contained in X^4,
and [Mas, Theorem 5.2], which removes the parameters at the expense of passing to X^8 (sharpening [Hru12, Lemma 4.5]).

Wagner conjectured that definable amenability is not required.
Hrushovski-Krupinski-Pillay give a counterexample.
This will go via the contrapositive of the following lemma.

Lemma. *Suppose H^{00} exists.*
Then $H^{000} \subseteq X^m$ for some m.

Proof. If H^{00} exists, then $H^{000} \leq H^{00}$.
But $H^{00} \subseteq X^m$ for some m. □

2 Identifying H^{000}

2.1 Thick relations and Lascar strong types

We recall some standard material on Lascar strong types.

An $\mathcal{L}$-formula $\phi(x, y)$ is thick if $\bigwedge_{i \neq j < \omega} \neg\phi(x_i, x_j)$ is inconsistent.

Lemma. *The thick formulas form a filter.*

Proof. If ϕ is thick and $\forall x, y.\ (\phi(x,y) \to \psi(x,y))$,
then ψ is thick.

Suppose $\phi \wedge \psi$ is not thick,
so say $(c_i)_i$ with $\vDash \neg\phi(c_i, c_j) \vee \neg\psi(c_i, c_j)$ for $i \neq j$;
colour according to the witnessing disjunct and apply Ramsey to show that either ϕ is not thick or ψ is not thick. □

Write $a \sim_1 b$ to mean that there exists an infinite indiscernible sequence $a, b, c_0, c_1, \ldots$.

Lemma. *$a \sim_1 b$ iff $\vDash \phi(a, b)$ for every thick ϕ.*
In particular, $\sim_1$ is $\bigwedge$-definable.

Proof. a, b extends to an infinite indiscernible sequence
iff $p := \mathrm{tp}(a, b)$ is in the EM-type of an infinite sequence
iff for no thick ϕ is $\neg\phi \in p$
iff p contains every thick ϕ. □

Let E_L be the transitive closure of $\sim_1$ ("same Lascar strong type").

Lemma. *E_L is the finest bounded invariant equivalence relation.*

Proof. E_L is invariant since $\sim_1$ is.

Let M be a model, and suppose $a \equiv_M b$.
Let p be a global coheir of $\mathrm{tp}(a/M)$,
and let $(c_i)_i$ be a Morley sequence in $p|_{Mab}$.
Then $a, c_0, c_1, \ldots$ and $b, c_0, c_1, \ldots$ are indiscernible,
so $a \sim_1 c_0 \sim_1 b$, so aE_Lb.

Hence $\equiv_M$ refines E_L, so the latter is bounded since the former is.

Now if E is a bounded invariant equivalence relation,
and $a \sim_1 b$,
then we can take an unboundedly long indiscernible sequence $a, b, \ldots$,
so we must have $a \sim b$.
Hence E_L refines E. □

2.2 Thick subsets and H^{000}

Definition. A definable symmetric subset $A \subseteq H$ is <u>thick</u> if

$$x, y \in X \to x^{-1}y \in A$$

is thick.

Lemma 2.1. *For $n \in \mathbb{N}$,*
$A \subseteq H$ is thick iff

$$x, y \in X^n \to x^{-1}y \in A$$

is thick.

Proof. Suppose $x, y \in X^n \to x^{-1}y \in A$ is not thick.
Say $a_i \in X^n$ and $a_i^{-1}a_j \notin A$ for any $i < j < \omega$.
Now $X^n \subseteq CX$ for finite C,
so by Ramsey an infinite subsequence is in some cX,
so translating we get $a'_i = c^{-1}a_{j_i} \in X$ with $a'^{-1}_i a'_j \notin A$ for $i < j < \omega$.
Hence $x, y \in X \to x^{-1}y \in A$ is not thick.

The converse is clear. $\square$

Let P be the intersection of the $\emptyset$-definable thick subsets of H.

Lemma 2.2. $P = \{a^{-1}b : a, b \in H,\ a \sim_1 b\}$.

Proof. Let $a, b \in H$ with $a \sim_1 b$.
Say $a, b \in X^n$.
If $A \subseteq H$ is thick,
then $\vDash a, b \in X^n \to a^{-1}b \in A$ so $a^{-1}b \in A$.
Hence $a^{-1}b \in P$.

Now $P' := \{a^{-1}b : a, b \in X,\ a \sim_1 b\}$ is $\bigwedge$-definable (since $\sim_1$ is) and symmetric, so say $P' = \bigwedge_k D_k$ where $D_k \subseteq H$ is symmetric definable.

Suppose D_k is not thick, say $c_i \in X$ and $c_i^{-1}c_j \notin D_k$ for $i < j < \omega$;
by Ramsey and compactness, we can assume $(c_i)_i$ is indiscernible.
Then $c_0 \sim_1 c_1$, so $c_0^{-1}c_1 \in P' \subseteq D_k$, contradiction.

So $P \subseteq P' \subseteq \{a^{-1}b : a, b \in H,\ a \sim_1 b\}$. $\square$

Proposition. $H^{000} = \langle P \rangle$.

Proof. $\langle P \rangle$ is invariant since P is.

We show $\langle P \rangle$ has bounded index.
Suppose $a, b \in H$ and aE_Lb.
Then say $a = a_0 \sim_1 a_1 \sim_1 \ldots \sim_1 a_n = b$ with $a_i \in H$.
Then $a_i^{-1}a_{i+1} \in P$ for each i by Lemma 2.2,
so $a^{-1}b \in \langle P \rangle$.
So since E_L is bounded, $\langle P \rangle$ has bounded index.

Now let $K \leq H$ be an invariant subgroup of bounded index.
Let $p \in P$.
By Lemma 2.2, $p = a^{-1}b$ for some a, b with $a \sim_1 b$.
The equivalence relation $x^{-1}y \in K$ is bounded invariant and aE_Lb,
so $p = a^{-1}b \in K$.
Hence $P \subseteq K$, so $\langle P \rangle \leq K$. $\square$

3 The example

$F_2 := \langle a, b \rangle$ = free group.

$$f : F_2 \to \mathbb{Z};\ a^{n_1} b^{n_2} \dots a^{n_{k-1}} b^{n_k} \mapsto \sum_i \operatorname{sgn}(n_i).$$

Fact 3.1. *f is a quasi-homomorphism: $f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$.*

Examples 3.2.

- $f(abab) = f(ab) + f(ab)$
- $f(abbab) = f(ab) + f(bab) - 1$
- $f(abb^{-2}ab) = f(ab) + f(b^{-2}ab) - 1$
- $f(abb^{-1}ab) = f(ab) + f(b^{-1}ab)$
- $f(abb^{-1}a^{-1}b) = f(ab) + f(b^{-1}a^{-1}b)$

Also $f(x^{-1}) = -f(x)$.

Work in monster model $(F_2^*; \mathbb{Z}^*; f) \succ (F_2; \mathbb{Z}; f)$.
So $X := \Gamma_f \subseteq F_2^* \times \mathbb{Z}^*$ is an approximate subgroup,
$X^{n+1} \subseteq (e, [-n, n]) \cdot X$.
$H := \langle X \rangle = (e, \mathbb{Z}) \cdot X$.

Let $A \subseteq H$ be thick.
Then $(a^k, 0) \in A$ for some $k > 0$.
Indeed, using symmetry of A,
otherwise $(a^k, 0)^{-1} \cdot (a^l, 0) = (a^{l-k}, 0) \notin A$ for any $k \neq l$, contradicting thickness.
Similarly, $(b^l, 0) \in A$ for some $l > 0$.
Note that $f((a^k b^l)^n) = 2n$ for $n \in \mathbb{N}$.

By saturation, we find $(s, 0), (t, 0) \in P = \bigwedge \{A : A \text{ thick}\}$,
with $f(s) = 1 = f(t)$ and $f((st)^n) = n$ for $n \in \mathbb{N}$.

So for $n \in \mathbb{N}$,
$((st)^n, 0) = ((s, 0), (t, 0))^n \in \langle P \rangle = H^{000}$,
but $((st)^n, 0) = ((st)^n, n) - (0, n) \notin X^n$.
So $H^{000} \not\subseteq X^n$.
So H^{00} does not exist.

References

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