Lec 15

$$\cancel{\frac{d}{dx}\int_a^x} \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big]_a^b$$

Remark (off syllabus):

The functions we've been working with - those built from x^α, e^x, $\ln x$ by adding, multiplying, composing, dividing, taking roots (e.g. $xe^{\sqrt{\ln(x+x^3)}}$) (the "elementary functions") have derivatives which are also of this form

We developed techniques to see a given such function as the derivative of another.

By the FTC any such fn has antiderivatives (on intervals where continuous)

Natural question: do elementary functions always have elementary antiderivatives

Answer: No

Pg Examples include interesting functions

e.g. "bell-curve"

e^{-x^2} does not have elementary antiderivative

also e^{x^2}, $x^2 e^{x^2}$, $x^4 e^{x^2}$

$\sqrt{x^3+1}$

Luckily, many natural interesting functions do have elementary antiderivatives (e.g. $\int x^2\,dx = \frac{x^3}{3} + c$)

so can efficiently calculate values and manipulate algebraically.

How to calculate $\int_a^b f(x)\,dx$ (assume $f(x)$ is continuous on $a \le x \le b$)

e.g. $\int_0^2 x e^{x^2}\,dx$ $\int_0^2 e^{x^2}\,dx$

Approach 1a: Use FTC + techniques for indefinite integrations.

Find (a formula for) $\int f(x)\,dx$, an antiderivative

by FTC $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$

e.g. $\int_0^2 x e^{x^2}\,dx = \left[\int x e^{x^2}\,dx\right]_a^b$ $u = x^2$

$= \left(\frac{1}{2}\int e^u\,du\right)\Big]_{x=a}^{x=b}$ $\frac{du}{dx} = 2x$

$$= \left(\frac{1}{2}e^u\right)\Big]_{x=a}^{x=b}$$

$$= \frac{1}{2}e^{x^2}\Big]_{x=a}^{x=b}$$

$$= \frac{1}{2}\left(e^{b^2} - e^{a^2}\right)$$

$$= \frac{1}{2}\left(e^4 - 1\right) \qquad (b=2,\ a=0)$$

$$\int_0^2 e^{x^2}\,dx = \int e^{x^2}dx\Big]_0^2$$

= no formula! Stuck!

Approach 1b: Use techniques for definite integration

- Later today, we'll develop versions of substitution parts etc which apply directly to definite integrals

we'll justify:

e.g. $\displaystyle\int_0^2 xe^{x^2}\,dx = \frac{1}{2}\int_0^4 e^u\,du \qquad u = x^2$

$$= \frac{1}{2}\left[e^u\right]_0^4$$

$$= \tfrac{1}{2}\left(e^4 - 1\right)$$

still no use for $\displaystyle\int_0^2 e^{x^2}dx$!

Approach 2: Numerical integration:

Estimate the integral, e.g. by evaluating for large n the Rièmann sum ~~$\Delta x \sum f(x$~~

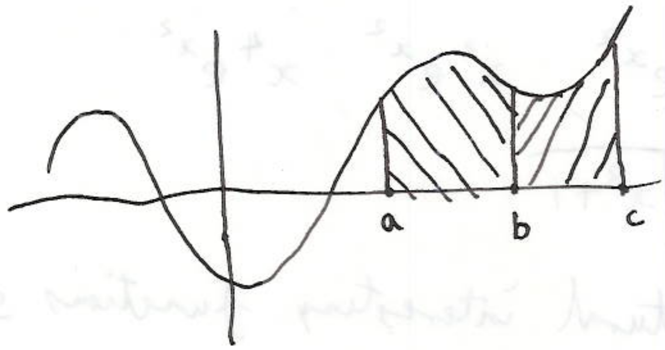
$$\frac{b-a}{n}\cdot\sum_{i=1}^{n} f(x_i)$$

(or similar Techniques, e.g. Simpson's rule)

Techniques of Definite Integration

Basic Rules:

(i) $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



(ii) $\displaystyle\int_a^a f(x)\,dx = 0$

(iii) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

(in fact (i) & (ii) imply (iii):

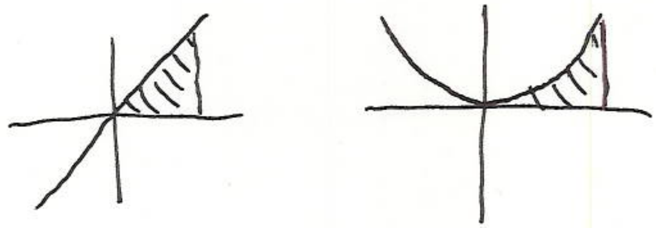
$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx \quad \text{(by (i))}$$
$$= 0 \quad \text{(by (ii))}$$

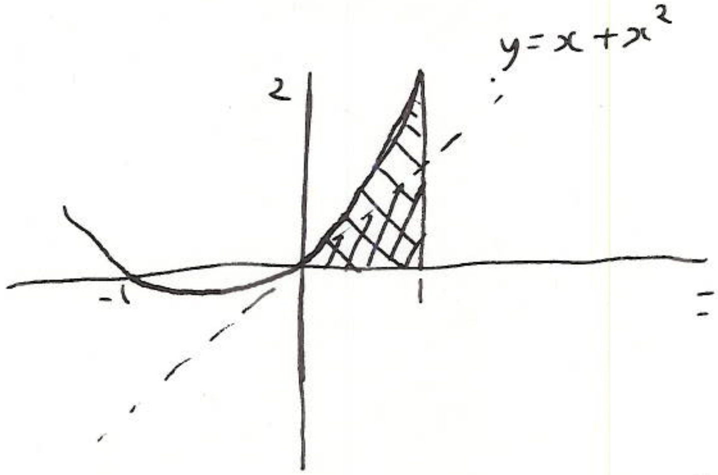
linearity

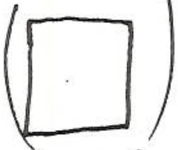
(iv) $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

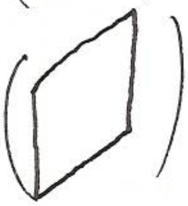
(v) $\displaystyle\int_a^b \lambda f(x)\,dx = \lambda\int_a^b f(x)\,dx$ (λ any real number)

e.g. $\displaystyle\int_0^1 x\,dx + \int_0^1 x^2\,dx = \int_0^1 (x+x^2)\,dx$





Area ()

= Area ()

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