Lec 16

Definite Integration by Parts:

If $f(x)$ and $g(x)$ functions with continuous derivatives $f'(x)$ and $g'(x)$ on $a \le x \le b$

then $$\int_a^b f(x)g'(x)\,dx = \left.\int f(x)g'(x)\,dx\right]_a^b \quad \text{(FTC)}$$

$$= \left(f(x)g(x) - \int f'(x)g(x)\,dx\right)\Big]_a^b \quad \text{(indefinite parts)}$$

$$= \left[f(x)g(x)\right]_a^b - \left[\int f'(x)g(x)\,dx\right]_a^b$$

$$= \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx \quad \text{(FTC)}$$

So we've deduced:

$$\boxed{\int_a^b f(x)g'(x)\,dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)\,dx}$$

e.g. $$\int_1^2 xe^x\,dx = \left[xe^x\right]_1^2 - \int_1^2 e^x\,dx$$

$f(x) = x \qquad g(x) = e^x$

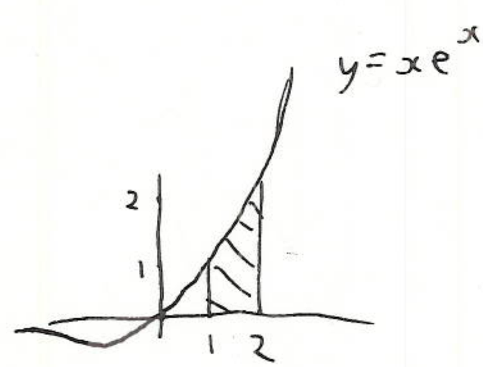
$f'(x) = 1 \qquad g'(x) = e^x$

$$= (2e^2 - e) - \int_1^2 e^x\,dx$$

$$= (2e^2 - e) - \left[e^x\right]_1^2$$

$$= (2e^2 - e) - (e^2 - e)$$

$$= e^2$$



$f(x)\Big|_3^4 = f(x)\Big]_3^4$

$\parallel$

$\left(\left[f(x)\right]_3^4 = f(4) - f(3)\right.$

Definite integration by substitution

$$\int_a^b f(g(x))g'(x)\,dx = \left.\int f(g(x))g'(x)\,dx\right]_a^b \qquad (FTC)$$

$$= \left[\int f(u)\frac{du}{dx}\,dx\right]_{x=a}^{x=b} \qquad u = g(x), \quad \frac{du}{dx} = g'(x)$$

$$= \left[\int f(u)\,du\right]_{x=a}^{x=b}$$

$$= \left[\int f(u)\,du\right]_{u=g(a)}^{u=g(b)}$$

$$= \int_{g(a)}^{g(b)} f(u)\,du \qquad (FTC)$$

So $$\boxed{\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$ where $u = g(x)$

e.g. $$\int_{1/3}^{8} x^2\sqrt{x^3+1}\,dx = \int_{1/3}^{8}\sqrt{u}\,x^2\cdot\tfrac{1}{3}3\,dx \qquad u = x^3+1, \quad \frac{du}{dx} = 3x^2$$

$$= \frac{1}{3}\int_{1/3}^{8}\sqrt{u}\,3x^2\,dx$$

$$= \frac{1}{3}\int_{1/3}^{8}\sqrt{u}\,\frac{du}{dx}\,dx$$

$$= \frac{1}{3}\int_{(1/3)^3+1}^{8^3+1}\sqrt{u}\,du = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_{28/27}^{513}$$

$$= \frac{2}{9}\left(513^{3/2} - \left(\frac{28}{27}\right)^{3/2}\right)$$

e.g. $\int_{-1}^{1} x^3 e^{x^2}\,dx$ $\qquad u = x^2$

$= \frac{1}{2}\int_{-1}^{1} u e^u \frac{du}{dx}\,dx \qquad \frac{du}{dx} = 2x$

$= \frac{1}{2}\int_{1}^{1} u e^u\,du$

$= 0$

Average value:

Recall: The average (more precisely, the "mean") of finitely many values $x_1, \dots, x_n$ (e.g. average student debt of my audience) is the sum divided by the number of values

$$V = \frac{1}{n}(x_1 + \dots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

In other words, it's that V such that the sum of the differences is 0

$$\sum (V - x_i) = (V - x_1) + \dots + (V - x_n) = 0$$

Average of a continuous function:

e.g. $T(t) :=$ temperature in $^\circ C$ of this room at time t seconds after midnight ~~this today~~ yesterday

What is the average temperature in the day?

Generally: average of $f(x)$ between a and b (f continuous on $a \le x \le b$)

<u>Idea</u>: can estimate average by taking n evenly spaced samples and calculating their average $V_n := \frac{1}{n} \sum_{i=1}^{n} f(x_i)$

can express as a Riemann sum:

$$\Delta x = \frac{b-a}{n}$$

$$\text{so } V_n = \cancel{\frac{1}{b-a} \Delta x \sum_{i=1}^{n} f(x_i)} \ \frac{1}{b-a} \Delta x \sum_{i=1}^{n} f(x_i)$$

<u>Define</u>: Average of f between a and b

$$:= \lim_{n \to \infty} V_n = \lim_{n \to \infty} \frac{1}{b-a} \Delta x \sum_{i=1}^{n} f(x_i)$$

$$= \frac{1}{b-a} \lim_{n \to \infty} \Delta x \sum_{i=1}^{n} f(x_i)$$

$$= \frac{1}{b-a} \int_a^b f(x)\, dx$$

(x_i is in the ith subinterval)