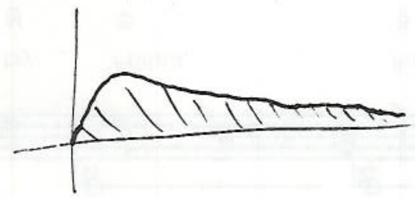
$$\left(\text{so } \int_0^\infty e^{-x}\,dx = \int_1^\infty \tfrac{1}{x^2}\,dx = 1 \right)$$

- $\displaystyle\int_0^\infty x e^{-x/2}\,dx$



Fact: if $c>0$, then for any p,

$$\boxed{\lim_{x\to\infty} x^p e^{-cx} = 0}$$

e.g. even $x^{1000000} e^{-x/7000} \xrightarrow[x\to\infty]{} 0$

(general rule of thumb,

"e^x beats x^p beats $\ln x$"

in terms of behaviour at ∞)

$$\int e^{-\frac{1}{2}x}\,dx = -2e^{-\frac{1}{2}x} + C$$

$$\int e^{-x/2}\,dx$$

$$\int_0^\infty x e^{-x/2}\,dx = \lim_{N\to\infty} \int_0^N x e^{-x/2}\,dx$$

$$= \lim_{N\to\infty} \left(\left[-2xe^{-x/2}\right]_0^N - \int_0^N -2e^{-x/2}\,dx \right)$$

$$= \lim_{N\to\infty} \left(-2Ne^{-N/2} - \left[4e^{-x/2}\right]_0^N \right)$$

$$= \lim_{N\to\infty} \left(-2Ne^{-N/2} - \left(4e^{-N/2} - 4\right) \right)$$

$$= -2\left(\lim_{N\to\infty} Ne^{-N/2}\right) - 4\left(\lim_{N\to\infty} e^{-N/2}\right) + 4$$

$$= 0 - 0 + 4$$

$$= 4$$

Numerical Integration (§6.3)

For many interesting functions, there is no antiderivative with a nice formula

e.g. e^{-x^2}, $\sqrt{x^3-1}$, $\frac{e^x}{x}$, ...

We can nonetheless *estimate* definite integrals of these functions $\int_a^b f(x)\,dx$

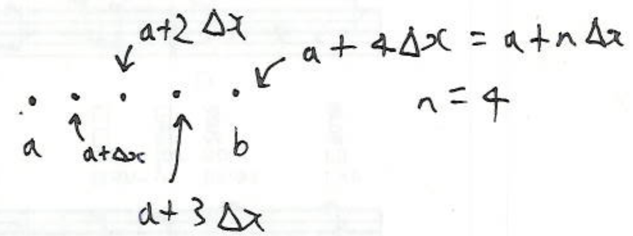
Method 1: Riemann sums

We *defined* $\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i)$

where $\Delta x = \frac{b-a}{n}$

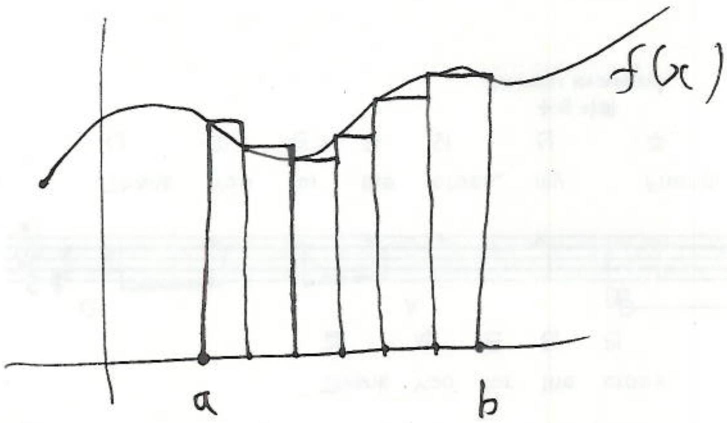
and x_i is in the i^{th} subinterval of size Δx

i.e. $a + (i-1)\Delta x \le x_i \le a + i\Delta x$



So if we set n to be large, and set $x_i := a + (i-1)\Delta x$

then $\int_a^b f(x)\,dx \approx \Delta x \sum_{i=1}^{n} f(x_i)$



Method 2 : Trapezoidal rule :

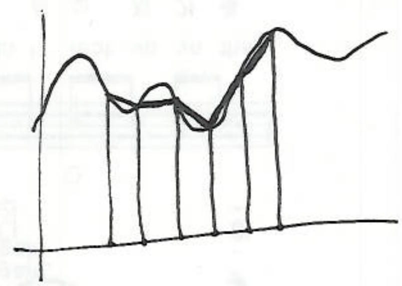
Idea:

Again, set $\Delta x = \frac{b-a}{n}$

and $x_i = a + (i-1)\Delta x \quad (i = 1, \ldots, n+1)$

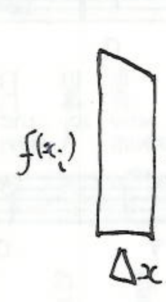
Then

$$\int_a^b f(x)\,dx \approx \left[\text{sum of areas of trapezoids}\right]$$



$$= \frac{\Delta x}{2}\left(f(x_1) + f(x_2)\right) + \frac{\Delta x}{2}\left(f(x_2) + f(x_3)\right) + \ldots + \frac{\Delta x}{2}\left(f(x_n) + f(x_{n+1})\right)$$

Area $= \frac{\Delta x}{2}\left(f(x_i) + f(x_{i+1})\right)$



$$\boxed{= \frac{\Delta x}{2}\left(f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right)}$$

Method 3 : Simpson's Rule :

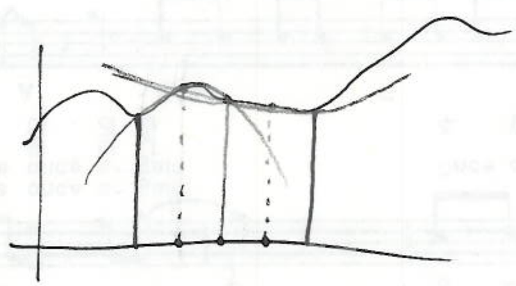
If n is even, $x_i = a + (i-1)\Delta x$, $\Delta x = \frac{b-a}{n}$

$(i = 1, \dots, n+1)$

$$\int_a^b f(x)\,dx = \frac{\Delta x}{3}\Big(f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + 2f(x_5) + \dots + 4f(x_n) + f(x_{n+1})\Big)$$

Generally <u>better</u> than the trapezoidal rule, but less obvious why it works

why it works (roughly);



the unique quadratic $Q(x)$ s.t. $Q(x_i) = f(x_i)$ $i = 1, 2, 3$

has integral $\displaystyle\int_{x_1}^{x_3} Q(x)\,dx = \frac{\Delta x}{3}\big(f(x_1) + 4f(x_2) + f(x_3)\big)$

<u>Example</u>; $\displaystyle\int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2$

Use Simpson with $n = 10$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.1}{3}\left(\frac{1}{1} + 4\frac{1}{1.1} + 2\frac{1}{1.2} + \dots + 4\frac{1}{1.9} + \frac{1}{2}\right)$$

$= 0.6931502$

$\ln 2 = 0.6931472$