Functions of two variables

Definition: A function of two variables, $f(x,y)$, assigns to certain pairs of real numbers (x,y) a real number value $f(x,y)$.
The domain of f, $\mathrm{Dom}(f)$, is the set of pairs (x,y) such that $f(x,y)$ is defined.

Examples:

(i) The distance in cm from a point in the real plane to a point which is x cm to the right and y cm up from the first point

is $r(x,y) = \sqrt{x^2+y^2}$

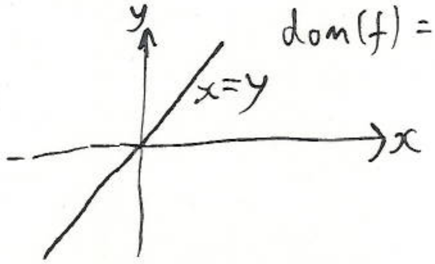
$\mathrm{dom}(r)$ is the whole real plane

(ii) The function given by the formula

$$f(x,y) = \frac{x+y}{x-y} \qquad \text{so } f(5,3) = \frac{5+3}{5-3} = \frac{8}{2} = 4$$

The domain of f is those (x,y) such that $x-y \neq 0$

i.e. those (x,y) such that $x \neq y$



$\mathrm{dom}(f)$ = whole real plane apart from the line $x=y$

(iii) We've seen functions of one variable which involve a constant

e.g. e^{rt} (if a quantity grows exponentially at ~~an~~ a rate r

(e.g. population growth

debt / savings (r is the "continuous interest rate")

then after t time, the quantity has grown by a factor of e^{rt})

We can see this as a function of 2 variable

$G(r,t) = e^{rt}$

(dom G = whole real plane

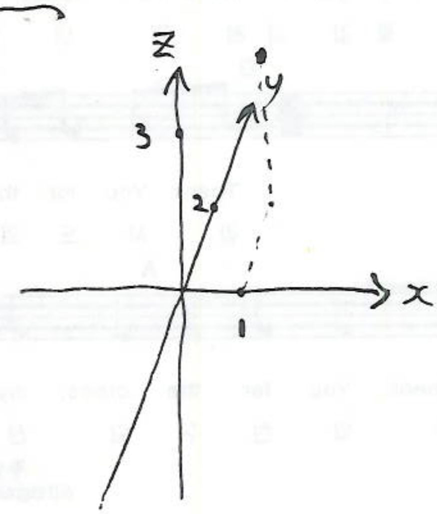
(negative r $\rightsquigarrow$ exponential decay

$r=0$ $\rightsquigarrow$ constant

negative t makes sense))

<u>Graphing functions of two variables</u>

If we fix axes, we can refer to points in space as triples (x, y, z) of real numbers



Then the graph of a function $f(x,y)$

is the set of points $(x, y, f(x,y))$

"Slicing vertically":

Given $f(x,y)$

If we fix (say) y to be a given real number

$y = a$

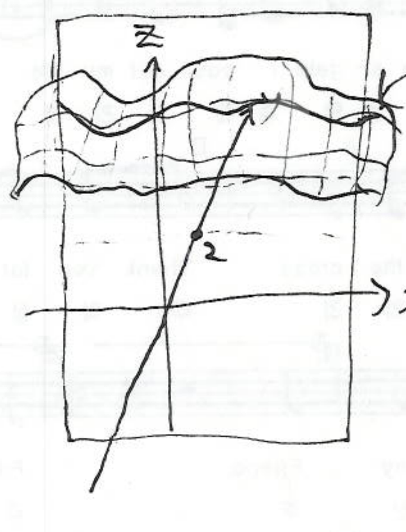
we get a function of 1 variable

$f(x,a)$

e.g. $r(x,y) = \sqrt{x^2+y^2}$

$r(x,1) = \sqrt{x^2+1}$

If we slice the graph of f with the plane $y=a$, we get the graph of $f(x,a)$



intersection of graph of f with plane $y=2$ is graph of $f(x,2)$

"Horizontal slicing"

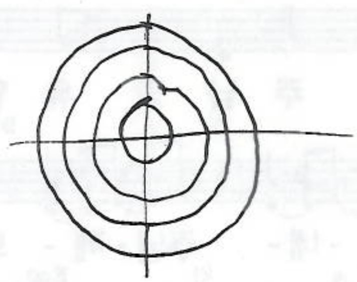
Cut the graph with the plane $z = a$

we get a subset of the plane
which is generally a curve "~~level curve~~
but not generally the graph of a function

as we vary a, we get a family of such curves
– the "level curves"

e.g. $r(x,y) = \sqrt{x^2+y^2}$

$r(x,y) = a \iff (x,y)$ are on the circle of radius a



http://www.math.mcmaster.ca/~mbays/teaching/1M03/notes/media/

Definition: The level curves of the function $f(x,y)$ are the curves defined by the equation

$$f(x,y) = a$$

e.g. $f(x,y) = e^{xy}$

Level curves are solutions to $e^{xy} = a$

$$e^{xy} = a \qquad (\log = \ln)$$

$$\Leftrightarrow \quad xy = \log(a)$$

$$\Leftrightarrow \quad y = \frac{\log(a)}{x} = \log(a)\frac{1}{x}$$

so graph of the function $\log(a)\frac{1}{x}$

Partial differentiation

$f(x,y)$

Definition:

The partial derivative of $f(x,y)$ with respect to x,

written $f_x(x,y)$

or $\frac{\partial}{\partial x} f(x,y)$

is the two-variable function

whose value at a point (x_0, y_0)

is the derivative at $x = x_0$ of the one-variable function $f(x, y_0)$

$\frac{\partial}{\partial x} f(x,y) = \frac{d}{dx} f(x,y)$ where on the right-hand side we think of $f(x,y)$ as a one-variable function where we treat y as a constant

$$\frac{\partial}{\partial y} f(x,y) = f_y(x,y)$$ is defined symmetrically (i.e. fix x, vary y)

e.g. $s(x,y) = x^2 + y^2$

$$\frac{\partial}{\partial x} s(x,y) = s_x(x,y)$$

$$= \frac{\partial}{\partial x}(x^2+y^2)$$

$$= 2x + 0$$

$$= 2x$$

$$s_y(x,y) = \frac{\partial}{\partial y}(x^2+y^2)$$

$$= 2y$$

Given $f(x,y)$

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y)$$

$\frac{\partial}{\partial x} f(a,b)$ is the rate at which $f(x,y)$ increases as we increase x starting from $x=a$, $y=b$

i.e. $$\frac{\partial}{\partial x} f(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

sim, $$\frac{\partial}{\partial y} f(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

e.g. $\dfrac{\partial}{\partial x} xy = y$

$\dfrac{\partial}{\partial y} xy = x$

$\dfrac{\partial}{\partial x} f(x,y) = f_x(x,y)$ is itself a function of two variables so it itself has derivatives!

We can consider

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y) \quad \text{and} \quad \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y)$$

Notation:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y) = \frac{\partial^2}{\partial x^2} f(x,y) = f_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y) = \cancel{f_{yx}(x,y)} \quad f_{xy}(x,y)$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial^2}{\partial x \partial y} f(x,y) = \cancel{f_{xy}(x,y)} \quad f_{yx}(x,y)$$

$$(f_y)_x = f_{yx}$$

So we have four second derivatives of f to play with:

$f_{xx}, f_{xy}, f_{yx}, f_{yy}$

So e.g. $\dfrac{\partial^2}{\partial x \partial y} f(x,y) =$ the rate of change with respect to y of the rate of change with respect to x of $f(x,y)$

$$\frac{\partial^2}{\partial x^2} \ln(x^2+y^2) = \frac{\partial}{\partial x}\frac{2x}{x^2+y^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} = 2(y^2 - x^2)$$

e.g. $f(x,y) = xy$

then $f_x = y$, $f_y = x$

$f_{xx} = 0$ $\quad f_{yy} = 0$

$f_{xy} = 1$ $\quad f_{yx} = 1$

e.g. $r(x,y) = \sqrt{x^2+y^2}$

$r_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$ $\qquad r_y = \frac{y}{\sqrt{x^2+y^2}}$

$r_{xx} = \frac{1}{\sqrt{x^2+y^2}} + \frac{-x\cdot 2x}{2(x^2+y^2)^{3/2}} \cancel{= \frac{x^2}{(x^2+y^2)^{3/2}}} \quad \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$

$\cancel{r_{yy} = \frac{y^2}{(x^2+y^2)^{3/2}}}$ $\qquad r_{yy} = \frac{x^2}{(x^2+y^2)^{3/2}}$

$r_{xy} = y\frac{-2x}{2(x^2+y^2)^{3/2}} = \frac{-xy}{(x^2+y^2)^{3/2}}$

$r_{yx} = x\frac{-2y}{2(x^2+y^2)^{3/2}} = \frac{-xy}{(x^2+y^2)^{3/2}}$

Fact: For "sufficiently nice" $f(x,y)$,

$f_{xy} = f_{yx}$