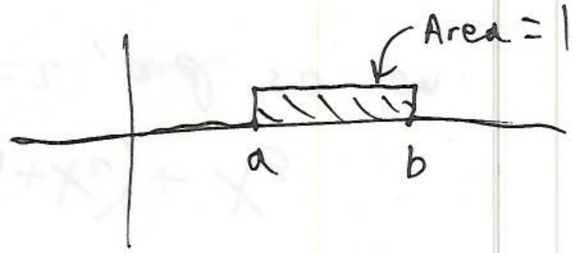
Examples of
Cont's Random Variables

~~Ex (i)~~

Uniformly Distributed ~~RVs~~ Random Variables:

X is uniformly distributed on the interval
$$a \le x \le b$$
if X takes values on the interval, all with equal

i.e. $$p_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{if } x < a \text{ or } x > b \end{cases}$$



e.g. The red light on my local pedestrian crossing lasts 55s

If I come to the crossing and find it's red,

X = time I have to wait

is distributed uniformly on $0 \le x \le 55$

e.g. $$P(5 \le X \le 10) = \int_5^{10} p_X(x)\,dx = \int_5^{10} \frac{1}{55}\,dx = \frac{10-5}{55} = 1/11$$

$$P(X \ge 30) = \int_{30}^{\infty} p_X(x)\,dx = \int_{30}^{55} p_X(x)\,dx + \int_{55}^{\infty} p_X(x)\,dx$$
$$= \int_{30}^{55} \frac{1}{55}\,dx + \int_{55}^{\infty} 0\,dx$$

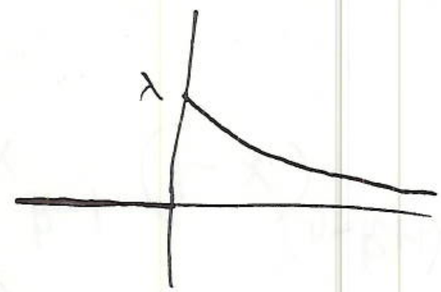
$$= \frac{55-30}{55}$$

$$= \frac{25}{55}$$

$$= 5/11$$

Exponential distributions:

$$p_X(x) := \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Example:

$X :=$ lifetime of a lightbulb in seconds (keep it on until it dies)

Why?

Idea (not on syllabus)

Let $D(t) := P(0 \leq X \leq t)$ = [probability it's dead by time t]

then $\cancel{p_X(x)}$ $p_X(t) = D'(t)$

$$= \lim_{h \to 0} \frac{D(t+h) - D(t)}{h}$$

$$= \lim_{h \to 0} \frac{P(t \leq X \leq t+h)}{h}$$

$= \lim_{h \to 0}$ [prob that it's is alive at time t] $\cdot$ [prob that it fails within h seconds after t] $\cdot \frac{1}{h}$

$$= \lim_{h \to 0} (1 - D(t)) \frac{\lambda h}{h} \qquad \text{some fixed } \lambda$$

$$= \lambda(1 - D(t))$$

solve this diff^{e} equation:

$$D' + \lambda D = \lambda$$

integrating factor: $e^{\int \lambda dx} = e^{\lambda x}$

general sol^{n}:

$$D(x) = \frac{1}{e^{\lambda x}} \int \lambda e^{\lambda x} dx = \frac{1}{e^{\lambda x}} (e^{\lambda x} + c)$$
$$= 1 + ce^{-\lambda x}$$

$D(0) = 0$

so $1 + c = 0$ so $c = -1$

so $D(x) = 1 - e^{-\lambda x}$

so $p_X(x) = D'(x) = \underline{\lambda e^{-\lambda x}}$.

Normal distributions — more next lecture

$$p_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

rescaled: $p_{(\sigma Z - \mu)}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x+\mu)^2/2\sigma^2}$

Expected Value:

$E(X)$ = expected value of X

= "average value of a load of samples of X"

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \, p_X(x) \, dx}$$

Examples: X uniform on $a \leq x \leq b$

$$E(X) = \cancel{\int_{-\infty}^{\infty} x \frac{1}{b-a}} \int_{-\infty}^{\infty} x \, p_X(x) \, dx$$

$$= \int_a^b x \frac{1}{b-a} \, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} \qquad b^2 - a^2 = (b+a)(b-a)$$

$$= \frac{b+a}{2}$$

= midpoint of the interval

Example: X has exponential distribution

$$p_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\, p_X(x)\, dx = \int_0^{\infty} x \lambda e^{-\lambda x}\, dx$$

$$= \lambda \left(\left[\frac{-x}{\lambda} e^{-\lambda x} \right]_0^{\infty} - \int_0^{\infty} \frac{-e^{-\lambda x}}{\lambda}\, dx \right)$$

$$= \left[\lambda \frac{-1}{\lambda^2} e^{-\lambda x} \right]_0^{\infty}$$

$$= \frac{0 - (-e^0)}{\lambda} = \frac{1}{\lambda}$$

[D20]

Example 7.3.4

X_1, X_2 independent $N(0,1)$

$$\Rightarrow Y = X_1 + X_2 \sim N(0, \underset{\sigma^2}{2})$$

proof - See text (Complete square)

Example 7.3.5 read in text

Theorem 7.3.2 merely extends previous results on sums when $n=2$ to general finite n. Use induction as in

$$X_1 + X_2 + X_3 = (X_1 + X_2) + X_3$$

to reduce $n=3$ to $n=2$, and so on.