For review session

Differential Equations — Separable — Linear

- $\int$ by parts — when to use
- Really awesome big $\int$ question
- Substitution

Variance and standard deviation

$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 p_X(x)\,dx \qquad \left(E(X) = \int_{-\infty}^{\infty} x\, p_X(x)\,dx\right)$$

$\sigma(X)$ = "standard deviation of X"

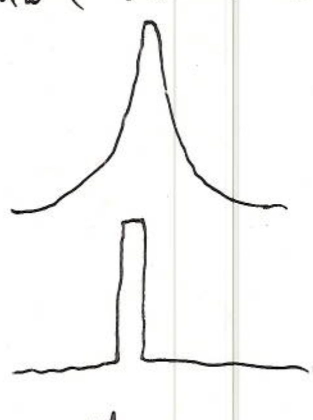
$$= \sqrt{Var(X)}$$

(i.e. if X has mean $E(X) = \mu$,

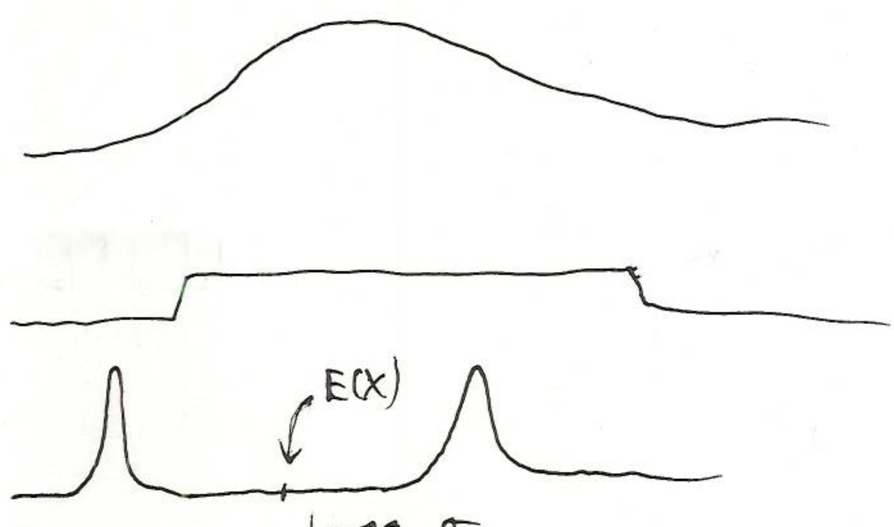
$$Var(X) = E((X-\mu)^2)$$

$$\sigma(X) = \sqrt{E((X-\mu)^2)} \quad)$$

Small standard deviation : very likely to be close to the mean



small σ



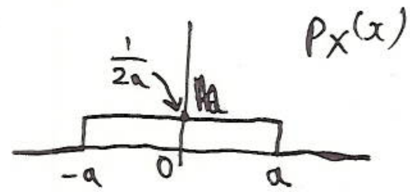
large σ

Remark: $\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 p_X(x)\,dx - E(X)^2$

Indeed:
$$\begin{aligned}
\mathrm{Var}(X) &= \int_{-\infty}^{\infty} (x - E(X))^2 p_X(x)\,dx \\
&= \int_{-\infty}^{\infty} (x^2 - 2E(X)x + E(X)^2)\, p_X(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx - 2E(X)\underbrace{\int_{-\infty}^{\infty} x\, p_X(x)\,dx}_{E(X)} + E(X)^2 \underbrace{\int_{-\infty}^{\infty} p_X(x)\,dx}_{1} \\
&= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx - 2E(X)^2 + E(X)^2 \\
&= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx - E(X)^2 .
\end{aligned}$$

Example: X uniformly distributed on $-a \le 0 \le a$

$E(X) = 0$



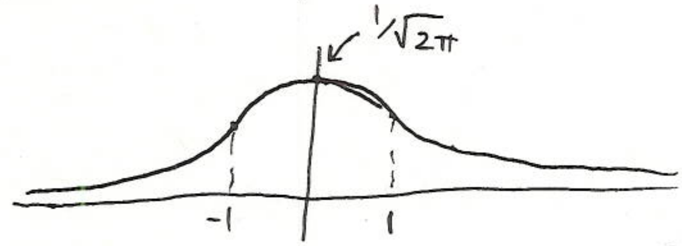
$$\begin{aligned}
\mathrm{Var}(X) &= \int_{-\infty}^{\infty} x^2 p_X(x)\,dx - 0^2 \\
&= \int_{-a}^{a} x^2 \frac{1}{2a}\,dx \\
&= \frac{1}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{1}{2a}\left(\frac{a^3 + a^3}{3}\right) = \frac{a^2}{3}
\end{aligned}$$

$\sigma(X) = a/\sqrt{3}$

Normal Distributions

The standard normal distribution has pdf

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



We say that X is normally distributed if

$X = \sigma Z + \mu$ for some $\sigma > 0$ and some μ
where Z has standard normal distribution

Recall: $e^{-x^2/2}$ was one of our basic examples
of a function which we "can't integrate"
(i.e don't have a nice formula for an antiderivative)

Nature's Cruel Trick!

Normal distributions are very common
(Central limit theorem:
resulting from averaging lots factors
"any sufficiently complicated random process
is roughly normally distributed")

Examples: X = height of a 20 year old male human
X = weight " " " " " "
X = IQ of a human adult

X = lifespan of a human

X = actual length of a ruler that's meant ~~to~~ be 30cm long

Suppose Z has standard normal distribution

i.e. $p_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

<u>Fact</u>:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} x e^{-x^2/2}\,dx = 0 \qquad \Rightarrow E(Z) = 0$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx = \sqrt{2\pi}$$

so $\mathrm{Var}(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx - E(Z)^2$

$= 1$

so $\sigma(Z) = 1$