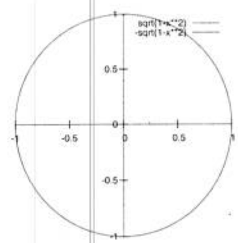
Midterm remarks: Review sessions **tonight** - see yellow website.
No table of integrals will be given: learn the table on p398 of Stewart.
ko na tcica bilma



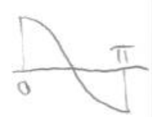
Examples: Area of the unit circle: We can think of this as the area enclosed by the graphs of $\sqrt{1-x^2}$ and $-\sqrt{1-x^2}$. So:

$$
\begin{aligned}
C &= \int_{-1}^{1} \left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right) dx \\
&= 2\int_{-1}^{1} \sqrt{1-x^2}dx \\
&= 2\int_{-1}^{1} \sqrt{1-\left(\cos\left(\arccos\left(x\right)\right)\right)^2}dx \\
&= 2\int_{-1}^{1} \sqrt{1-\cos\left(\theta\right)^2}dx \\
&= 2\int_{-1}^{1} \sin\left(\theta\right) dx \\
&= 2\int_{-1}^{1} -\sin^2\left(\theta\right) \underbrace{\frac{1}{-\sin\left(\theta\right)}}_{\frac{d\theta}{dx}} dx \\
&= 2\int_{\pi}^{0} -\sin^2\left(\theta\right) d\theta \\
&= 2\int_{0}^{\pi} \sin^2\left(\theta\right) d\theta \\
&= \int_{0}^{\pi} \left(1-\cos\left(2\theta\right)\right) d\theta \\
&= \left(\left[\theta\right]_0^{\pi} - \frac{1}{2}\left[\sin\left(2\theta\right)\right]_0^{\pi}\right) \\
&= \pi - 0 \\
&= \pi
\end{aligned}
$$

$$\left(\theta = \arccos\left(x\right);\ \frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}} = \frac{1}{-\sin\left(\theta\right)}\right)$$

substitution rule

$$\arccos'(\cos\theta) = \frac{1}{\cos'\theta} = \frac{-1}{\sin\theta} = \frac{-1}{\sqrt{1-\cos^2\theta}}$$



$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

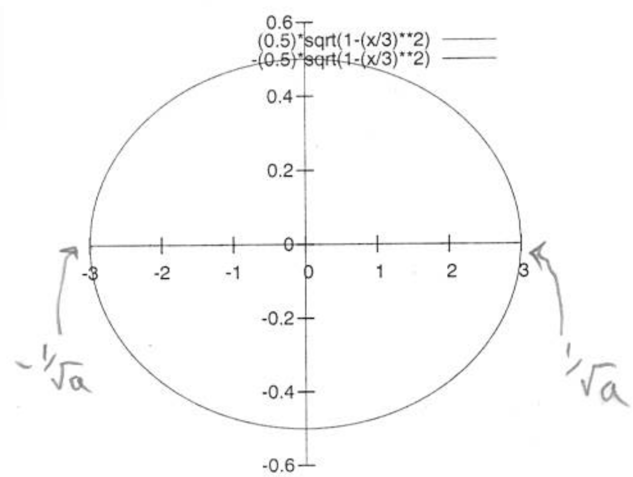
Area of an ellipse: After rotating and translating to the origin, any ellipse can be represented as the solutions to $ax^2 + by^2 = 1$, i.e.

$$y = \pm \frac{1}{\sqrt{b}} \sqrt{1 - ax^2}.$$

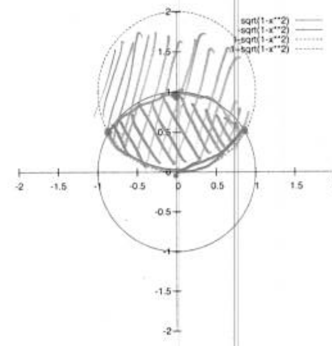
So the area is

$$\begin{aligned}
E_{b,a} &= \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \left(\frac{1}{\sqrt{b}} \sqrt{1 - ax^2} - \left(-\frac{1}{\sqrt{b}} \sqrt{1 - ax^2} \right) \right) dx \\
&= 2 \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \frac{1}{\sqrt{b}} \sqrt{1 - ax^2} dx \\
&\cancel{= \frac{2}{\sqrt{b}} \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \sqrt{1 - ax^2} dx} \\
&= \frac{2}{\sqrt{b}} \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \sqrt{1 - ax^2} dx = \frac{2}{\sqrt{b}\sqrt{a}} \int_{-\frac{1}{\sqrt{a}}}^{\frac{1}{\sqrt{a}}} \sqrt{1 - ax^2} \sqrt{a}\, dx \quad \left(u = (\sqrt{a})x; \frac{du}{dx} = \sqrt{a} \right) \\
&\qquad u^2 = (\sqrt{a})^2 x^2 = ax^2 \\
&\qquad du = \frac{du}{dx} dx \\
&= \frac{2}{\sqrt{b}\sqrt{a}} \int_{-1}^{1} \sqrt{1 - u^2}\, \cancel{dx}\, du \\
&= \frac{\pi}{\sqrt{ab}},
\end{aligned}$$

which note makes a lot of sense: $ax^2 + by^2 = 1 \leftrightarrow (\sqrt{a}x)^2 + \left(\sqrt{b}y\right)^2 = 1$, so our ellipse is a circle scaled horizontally by $\frac{1}{\sqrt{a}}$ and vertically by $\frac{1}{\sqrt{b}}$.



Area of a crescent:
As seen from the earth, the disc of the sun has approximately the same radius as the disc of the moon. During a solar eclipse, the latter slides over the former. When the disc of the moon is centred at the edge of the disc of the sun, what proportion of the sun's disc is covered?



$\sqrt{1-x^2} = -\sqrt{1-x^2}+1$

Intersection points:

$$\sqrt{1-x^2} = 1-\sqrt{1-x^2} \leftrightarrow 2\sqrt{1-x^2} = 1 \leftrightarrow x = \pm\frac{\sqrt{3}}{2}$$

Area of covered area:

$$\begin{aligned}
C &= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-x^2} - \left(1-\sqrt{1-x^2}\right)\right) dx \\
&= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(2\sqrt{1-x^2} - 1\right) dx \\
&= 2\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}dx - \sqrt{3} \\
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1-\cos(2\theta))\,d\theta - \sqrt{3} \\
&= [\theta]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{1}{2}[\sin(2\theta)]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \sqrt{3} \\
&= \frac{2\pi}{3} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \sqrt{3} \\
&= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} - \sqrt{3} \\
&= \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \\
&= 0.391\pi
\end{aligned}$$

$= [x]_{-\sqrt{3}/2}^{\sqrt{3}/2} = \sqrt{3}/2 - \left(-\sqrt{3}/2\right) = \sqrt{3}$

So 0.391 of the sun's disc is blocked.

Volumes

Example - Volume of a sphere: Consider a sphere of radius r, centred at the origin (0,0,0).

Chop it perpendicular to the x-axis into n slivers of equal width.

The volume of the sphere is the sum of the volumes of the slivers.

For large n, i.e. for thin slivers, each sliver is roughly a cylinder of width $\Delta_n = \frac{2r}{n}$. The radius depends on x: the i^{th} sliver has radius $\sqrt{r^2 - (x_i^*)^2}$ on its right face, where $x_i^* = -r + i\Delta_n$.

So we can estimate the volume of the i^{th} sliver as

$$\Delta_n \pi \left(\sqrt{r^2 - (x_i^*)^2}\right)^2 = \Delta_n \pi \left(r^2 - (x_i^*)^2\right)$$

So our estimate for the volume with n slivers is

$$V_n = \sum_{i=1}^{n} \Delta_n \pi \left(r^2 - (x_i^*)^2\right).$$

As $n \to \infty$, our estimates converge to the actual volume. So the volume of the sphere is

$$\begin{aligned} V &= \lim_{n\to\infty} V_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_n \pi \left(r^2 - (x_i^*)^2\right) \\ &= \int_{-r}^{r} \pi \left(r^2 - x^2\right) dx \\ &= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\ &= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 - \frac{-r^3}{3}\right)\right) \\ &= \frac{4\pi r^3}{3}. \end{aligned}$$

General formula: If a shape lies between $x = a$ and $x = b$, and the area of a cross-section perpendicular to the x-axis is a continuous function $A(x)$, then the volume is

$$\int_a^b A(x)\, dx.$$

$A(x) = \pi(r^2 - x^2)$ for sphere of radius r

Indeed, the argument above indicates that the volume is

$$\lim_{n\to\infty} \sum_{1}^{n} A(x_i^*)\, \Delta_n$$

$\Delta_n = \frac{b-a}{n}$

where we divide $[a, b]$ into n intervals of equal width, and x_i^* is a point in the i^{th} interval and Δ_n is the width of an interval. But this is precisely the definition of the integral $\int_a^b A(x)\, dx$, which exists since A is continuous.