Integration by parts

$$\frac{d}{\mathrm{dx}} f(x)\, g(x) = f'(x)\, g(x) + f(x)\, g'(x)$$

$f(x)g(x) = \int f'(x)g(x) + f(x)g'(x)dx + C$

so

$$\int \left(f'(x)\, g(x) + f(x)\, g'(x)\right) dx = f(x)\, g(x) + C$$

so

$$\boxed{\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx.}$$

"$\int f'g = fg - \int fg'$"

Example: $(f(x) = \sin x \quad g(x) = x)$

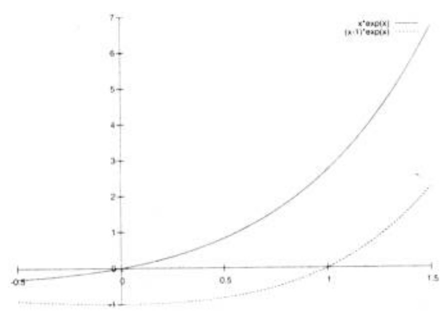
$$\int x \cos(x)\, dx = \int \sin'(x)\, x dx = \sin(x)\, x - \int \sin(x)\, 1 dx = x \sin(x) + \cos(x) + C.$$

Remark: Setting $u = f(x)$ and $v = g(x)$ (and assuming u is a function of v and v is a function of u) we can use the Substitution rule to rewrite this as

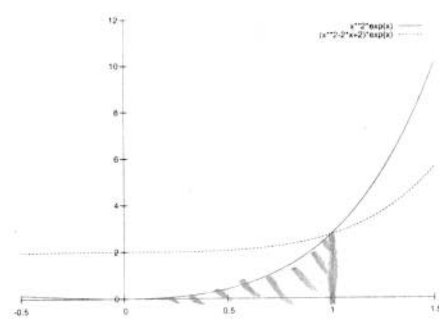
$$\int \underset{g(x)}{v}\, \underset{f'(x)dx}{\mathrm{du}} = uv - \int \underset{f(x)}{u}\, \underset{g'(x)dx}{\mathrm{dv}}.$$

Examples:

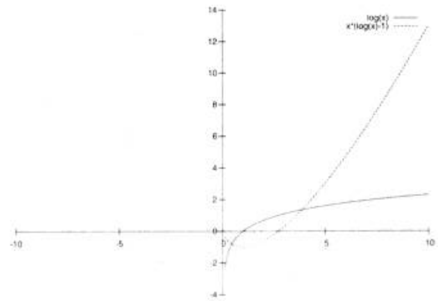
- $\int x e^x dx$



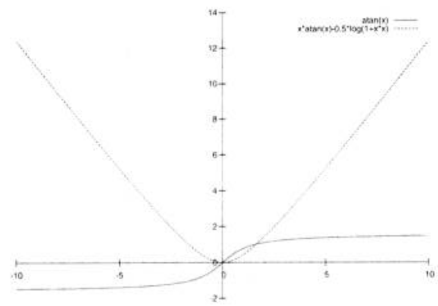
- $\int_0^1 x^2 e^x dx$



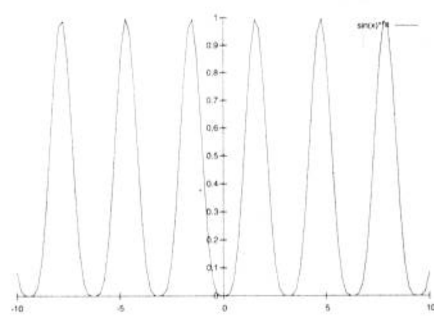
- $\int \ln x dx$



- $\int \arctan x dx$



- $\int \sin^4 x dx$



(Recall: $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$)

- $\int x e^x \, dx$ $\qquad g(x) = x \quad g'(x) = 1$

$\qquad\qquad\qquad\quad f(x) = e^x \quad f'(x) = e^x$

$\quad \parallel$

$\int f'(x) g(x) \, dx$

$\quad \parallel$

$f(x) g(x) - \int f(x) g'(x) \, dx$

$= x e^x - \int e^x \cdot 1 \, dx$

$= x e^x - e^x + C$

$= (x-1) e^x + C$

<u>Check</u>: $\frac{d}{dx}(x e^x - e^x + C)$

$= x e^x + e^x - e^x$

$= x e^x$ ✓

- $\int_0^1 x^2 e^x \, dx =$

$\qquad [x^2 e^x]_0^1 - \int_0^1 2 x e^x \, dx$

$= [x^2 e^x]_0^1 - 2 [(x-1) e^x]_0^1$

$= e - 2((1-1)e - (0-1)e^0)$

$= e - 2$

$\int f' g = f g - \int f g'$

$g(x) = x^2 \quad g'(x) = 2x$

$f(x) = e^x \quad f'(x) = e^x$

$\boxed{\text{Definite integrals by parts:} \quad \int_a^b f'(x) g(x) \, dx = [f(x) g(x)]_a^b - \int_a^b f(x) g'(x) \, dx}$

$$\cdot \int \ln x \, dx$$

$f(x) = x \quad f'(x) = 1$

$g(x) = \ln x \quad g'(x) = 1/x$

$$= \int 1 \cdot \ln x \, dx$$

$$= x \ln x - \int x \frac{1}{x} \, dx$$ by parts

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + c$$

$$= x(\ln x - 1) + c$$

check, $\frac{d}{dx} x(\ln x - 1)$

$$= (\ln x - 1) + x \frac{1}{x}$$

$$= \ln x - 1 + 1$$

$$= \ln x$$

$$\cdot \int \arctan(x) \, dx = \int 1 \cdot \arctan(x) \, dx$$

$$= x \arctan(x) - \int x \arctan'(x) \, dx$$

$$= x \arctan(x) - \int \frac{x}{1+x^2} \, dx$$

$$= \quad " \quad - \frac{1}{2} \int u^{-1} \, du$$

$u = 1 + x^2$

$\frac{du}{dx} = 2x$

$$= \quad " \quad - \frac{1}{2} \ln|u| + c$$

$$= x \arctan(x) - \frac{\ln(1+x^2)}{2} + c$$

check; $\frac{d}{dx} \left(\quad " \quad " \quad \right)$

$$= \arctan(x) + \frac{x}{1+x^2} - \frac{1}{2} 2x \frac{1}{1+x^2}$$

$$= \arctan(x) \checkmark$$