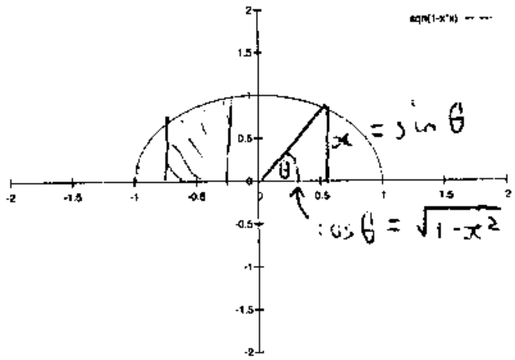
Trigonometric substitution

Consider (again) $\int \sqrt{1-x^2}\,\mathrm{dx}$.



Note this only makes sense for x in $[-1,1]$.
Recall Pythagoras.
So if $x = \sin(\theta)$,

$$\int \sqrt{1-x^2}\,\mathrm{dx} = \int \cos(\theta)\,\mathrm{dx}\,.$$

To get an integral involving only θ, we can use the substitution rule backwards. $\frac{\mathrm{dx}}{\mathrm{d}\theta} = \cos(\theta)$, so by the substitution rule

$$\int \cos(\theta)\cos(\theta)\,\mathrm{d}\theta = \int \sqrt{1-x^2}\,\frac{\mathrm{dx}}{\mathrm{d}\theta}\,\mathrm{d}\theta$$
$$= \int \sqrt{1-x^2}\,\mathrm{dx}.$$

$x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

So

$$\begin{aligned}\int \sqrt{1-x^2}\,\mathrm{dx} &= \int \cos(\theta)\cos(\theta)\,\mathrm{d}\theta\\ &= \int \cos^2(\theta)\,\mathrm{d}\theta\\ &= \frac{1}{2}\int (1+\cos(2\theta))\,\mathrm{d}\theta \qquad = \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C\\ &= \frac{1}{2}\left(\arcsin(x) + \sin(\theta)\cos(\theta)\right) + C\\ &= \frac{1}{2}\left(\arcsin(x) + x\sqrt{1-x^2}\right) + C.\end{aligned}$$

recall range(arcsin) $= [-\frac{\pi}{2}, \frac{\pi}{2}]$

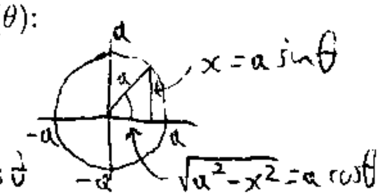
So e.g. we recover

$$\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{dx} = \frac{1}{2}\left(\frac{\pi}{2} + 1\sqrt{1-1^2} - \left(-\frac{\pi}{2} + (-1)\sqrt{1-(-1)^2}\right)\right) = \frac{\pi}{2}.$$

$\int \sqrt{c-x^2}\,\mathrm{dx}$ **with** c **positive:** Let $a = \sqrt{c}$, so $\sqrt{c-x^2} = \sqrt{a^2-x^2}$, think about a circle of radius a, and do inverse substitution with $x = a\sin(\theta)$:

$$\int \underbrace{a\cos(\theta)}_{\sqrt{a^2-x^2}}\,\underbrace{a\cos(\theta)}_{\frac{dx}{d\theta}}\,\mathrm{d}\theta = \int \sqrt{a^2-x^2}\,\mathrm{dx}$$

$x = a\sin\theta \quad \frac{dx}{d\theta} = a\cos\theta$



Functions involving $\sqrt{a^2 - x^2}$: e.g.

$x = a \sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$\sqrt{a^2 - x^2} = a\cos\theta$

$$\int x^3\sqrt{a^2-x^2}\,\mathrm{dx} = \int (a\sin(\theta))^3\,(a\cos(\theta))\,(a\cos(\theta))\,\mathrm{d}\,\theta$$

$$\boxed{= a^5 \int \sin^3\theta \cos^2\theta \, d\theta}$$

$\sin^2\theta = 1 - \cos^2\theta$

$a^5 \int \sin\theta\,(\cos^2\theta - \cos^4\theta)\,d\theta$

$u = \cos\theta$

$\Rightarrow -a^5 \int (u^2 - u^4)\,du$

Definite integrals:

$b = a\sin\theta$

$b/a = \sin\theta$

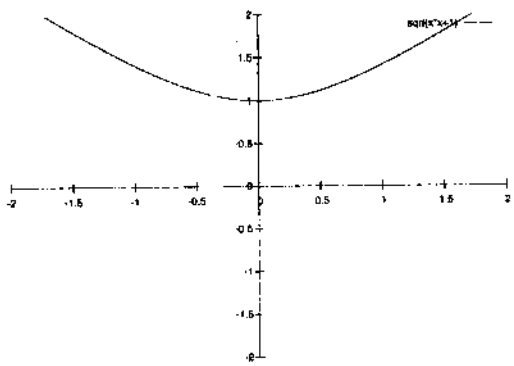
$\theta = \arcsin(b/a)$

$$\int_b^c x^3\sqrt{a^2-x^2}\,\mathrm{dx} = \int_{\arcsin(b/a)}^{\arcsin(c/a)} (a\sin(\theta))^3\,(a\cos(\theta))\,a\cos(\theta)\,\mathrm{d}\,\theta$$

where recall we chose arcsin to have range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

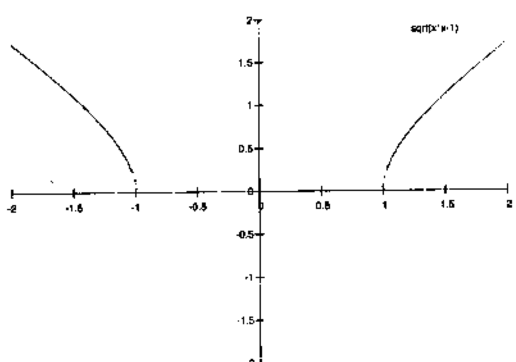
Handling other signs:

- $\sqrt{a^2 + x^2}$:



try inverse substitution $x = a\tan(\theta)$; so $\sqrt{a^2+x^2} = a\sec(\theta)$.
(Using $\tan^2 + 1 = \sec^2$)

- $\sqrt{x^2 - a^2}$:



try inverse substitution $x = a\sec(\theta)$; so $\sqrt{x^2-a^2} = a\tan(\theta)$.

- $\sqrt{-x^2 - a^2}$: never defined!

Examples:

- $\int_2^4 x^3\sqrt{x^2-4}\,\mathrm{dx}$
- $\int_2^4 x\sqrt{x^2-4}\,\mathrm{dx}$
- $\int x\,(x^2-2+\pi)^{\frac{3}{2}}\,\mathrm{dx}$

$\tan^2\theta + 1 = \sec^2\theta$

$(a\tan)^2 + a^2 = \cancel{(}\sec)^2$

$(a\tan\theta)^2 + a^2 = (a\sec\theta)^2$

$$\int_2^4 x^3\sqrt{x^2-4}\,dx$$

$x = 2\sec\theta$ $\qquad \sqrt{x^2-4} = 2\tan\theta$

$\frac{dx}{d\theta} = 2\tan\theta\sec\theta$

$$= \int_0^{\pi/3} (2\sec\theta)^3(2\tan\theta)(2\tan\theta\sec\theta)\,d\theta$$

$$= 2^5\int_0^{\pi/3} \tan^2\theta\sec^4\theta\,d\theta$$

$2 = 2\sec\theta$

$1 = \sec\theta$

$1 = \cos\theta$

$\theta = 0$

$4 = 2\sec\theta$

$2 = \sec\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \arccos\frac{1}{2}$

$= \pi/3$

$$= 2^5\int_0^{\pi/3} \tan^2\theta\,(\tan^2\theta+1)\sec^2\theta\,d\theta$$

$$= 2^5\int_0^{\tan(\pi/3)} u^2(u^2+1)\,du$$

$u = \tan\theta$

$\frac{du}{d\theta} = \sec^2\theta$