Examples:

- $\int_2^4 x^3\sqrt{x^2-4}\,\mathrm{dx}$
- $\int_2^4 x\sqrt{x^2-4}\,\mathrm{dx}$
- $\int x\,(x^2-2+\pi)^{\frac{3}{2}}\,\mathrm{dx}$

$\tan^2\theta + 1 = \sec^2\theta$

$(a\tan)^2 + a^2 = (a\sec)^2$

$(a\tan\theta)^2 + a^2 = (a\sec\theta)^2$

$x = 2\sec\theta \qquad \sqrt{x^2-4} = 2\tan\theta$

$\frac{dx}{d\theta} = 2\tan\theta\sec\theta$

- $\int_2^4 x^3\sqrt{x^2-4}\,dx$

$= \int_0^{\pi/3} (2\sec\theta)^3(2\tan\theta)(2\tan\theta\sec\theta)\,d\theta$

$2 = 2\sec\theta \qquad 4 = 2\sec\theta$

$1 = \sec\theta \qquad 2 = \sec\theta$

$1 = \cos\theta \qquad \cos\theta = 1/2$

$\theta = 0 \qquad \theta = \arccos 1/2 = \pi/3$

$= 2^5\int_0^{\pi/3} \tan^2\theta\sec^4\theta\,d\theta$

$\cancel{= 2^5\int_0^{\pi/3} \sec^2\theta\,(\tan^2\theta+1)\sec^2\theta\,d\theta}$

$\cancel{= 2^5\int_0^{\tan(\pi/3)} u^2(u^2+1)\,du}$

$\boxed{u = \tan\theta \qquad \frac{du}{d\theta} = \sec^2\theta}$

$= 2^5\int_0^{\pi/3} \tan^2\theta(\tan^2\theta+1)\sec^2\theta\,d\theta$

$= 2^5\int_0^{\tan(\pi/3)} u^2(u^2+1)\,du$

- $\int x\sqrt{x^2-4}\,dx = \frac{1}{2}\int\sqrt{u}\,du \qquad u = x^2-4 \qquad \frac{du}{dx} = 2x$

$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$

$= \frac{1}{2}\frac{(x^2-4)^{3/2}}{3/2} + C$

$\int x^3(x^2-2+\pi)^{3/2}\,dx$

$= \int x^3\left(\sqrt{x^2-2+\pi}\right)^3 dx$

$= \int x^3\left(\sqrt{x^2+(\sqrt{\pi-2})^2}\right)^3 dx$

$= \int (a\tan\theta)^3 (a\sec\theta)^3 a\sec^2\theta\, d\theta$

$= a^7 \int \tan^3\theta \sec^5\theta\, d\theta$

(use $u=\sec\theta$)

$= a^7\int(\sec^2\theta-1)\tan\theta\sec\theta\sec^4\theta\, d\theta$

$= a^7\int(u^2-1)u^4\, du$

$= a^7\left(\frac{u^7}{7}-\frac{u^5}{5}\right)+C$

$= a^7\left(\frac{\sec^7\theta}{7}-\frac{\sec^5\theta}{5}\right)$

$= a^7\left(\frac{(\sqrt{x^2+a^2})^7}{a^7\,7}-\frac{(\sqrt{x^2+a^2})^5}{5a^5}\right)+C$ using $\sec\theta=\frac{\sqrt{x^2+a^2}}{a}$

$\pi-2>0$

$\pi-2=(\sqrt{\pi-2})^2$

$a=\sqrt{\pi-2}$

$x=a\tan\theta$

$\frac{dx}{d\theta}=a\sec^2\theta$

$x^2+a^2=(a\sec\theta)^2$

$\sqrt{x^2+a^2}=a\sec\theta$

$a^2\tan^2\theta+a^2$

$=a^2(\tan^2\theta+1)$

$=a^2\sec^2\theta$

Partial fractions

Definition: A <u>rational function</u> is one of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.

e.g. $\frac{x^3-2x+1}{x^2-7}$, x^2, $\frac{1}{x^9+1}$

We already know how to integrate some of these.

Recall:

- $\int x^n \, \mathrm{dx} = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
- $\int \frac{1}{x} \, \mathrm{dx} = \ln|x| + C$
- $\int \frac{1}{x^2+1} \, \mathrm{dx} = \arctan x + C$
- $\int \frac{2x}{x^2+1} \, \mathrm{dx} = \int \frac{1}{u} du = \ln|x^2+1| + C$

$u = x^2+1$

$\frac{du}{dx} = 2x$

How about $\int \frac{1}{x^2-1} \, \mathrm{dx}$?

$$
\begin{aligned}
-\int \frac{1}{x^2-1} \, \mathrm{dx} &= -\int \frac{1}{(x+1)(x-1)} \, \mathrm{dx} \\
&= \int \frac{1}{2} \left(\frac{1}{x+1} - \frac{1}{x-1} \right) \mathrm{dx} \\
&= \frac{1}{2} \left(\int \frac{1}{x+1} \, \mathrm{dx} - \int \frac{1}{x-1} \, \mathrm{dx} \right) \\
&= \frac{1}{2} \left(\ln|x+1| - \ln|x-1| \right) + C \\
&= \ln \sqrt{\frac{|x+1|}{|x-1|}} + C
\end{aligned}
$$

$$\frac{1}{x+1} - \frac{1}{x-1} = \frac{(x-1)-(x+1)}{(x+1)(x-1)} = \frac{-2}{(x+1)(x-1)}$$

The key trick here was to recognise the rational function $\frac{-1}{(x+1)(x-1)}$ as being a linear combination of simpler rational functions ("partial fractions"), namely $\frac{1}{2}\left(\frac{1}{x+1} - \frac{1}{x-1}\right)$.

We will see that this technique, along with the integrals recalled above and polynomial division, allows us to integrate **any** rational function.

Example of using polynomial division:

$$
\begin{aligned}
\int \frac{x^3+1}{x^2+1} \, \mathrm{dx} &= \int \frac{x(x^2+1) - x + 1}{x^2+1} \, dx \\
&= \int \left(x + \frac{1-x}{x^2+1} \right) \mathrm{dx} \\
&= \int \left(x + \frac{1}{x^2+1} - \frac{x}{x^2+1} \right) \mathrm{dx} \\
&= \frac{x^2}{2} + \arctan(x) - \frac{\ln|x^2+1|}{2} + C
\end{aligned}
$$

$x^3+1 = (x^2+1)x - x + 1$

Fact I: Any polynomial over the reals can be factored as a product of linear and quadratic factors.