Partial fractions

Definition: A rational function is one of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials.

e.g. $\frac{x^3-2x+1}{x^2-7}, x^2, \frac{1}{x^9+1}$

We already know how to integrate some of these.

Recall:

- $\int x^n \, \mathrm{dx} = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$
- $\int \frac{1}{x} \, \mathrm{dx} = \ln|x| + C$
- $\int \frac{1}{x^2+1} \, \mathrm{dx} = \arctan x + C$
- $\int \frac{2x}{x^2+1} \, \mathrm{dx} = \int \frac{1}{u} du = \ln|x^2+1| + C$

How about $\int \frac{1}{x^2-1} \, \mathrm{dx}$?

$$\begin{aligned}
\int \frac{1}{x^2-1} \, \mathrm{dx} &= \int \frac{1}{(x+1)(x-1)} \, \mathrm{dx} \\
&= \int \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) \mathrm{dx} \\
&= \frac{1}{2}\left(\int \frac{1}{x-1} \, \mathrm{dx} - \int \frac{1}{x+1}\right) \mathrm{dx} \\
&= \frac{1}{2}\left(\ln|x-1| - \ln|x+1|\right) \\
&= \ln\sqrt{\frac{|x-1|}{|x+1|}}
\end{aligned}$$

The key trick here was to recognise the rational function $\frac{1}{(x+1)(x-1)}$ as being a linear combination of simpler rational functions ("partial fractions"), namely $\frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$.

We will see that this technique, along with the integrals recalled above and polynomial division, allows us to integrate **any** rational function.

Example of using polynomial division:

$$\begin{aligned}
\int \frac{x^3+1}{x^2+1} \, \mathrm{dx} &= \int \frac{x(x^2+1) - x + 1}{x^2+1} \\
&= \int \left(x + \frac{1-x}{x^2+1}\right) \mathrm{dx} \\
&= \int \left(x + \frac{1}{x^2+1} - \frac{x}{x^2+1}\right) \mathrm{dx} \\
&= \frac{x^2}{2} + \arctan(x) - \frac{\ln|x^2+1|}{2} + C
\end{aligned}$$

Fact I: Any polynomial over the reals can be factored as a product of linear and quadratic factors.

$$\frac{x^2-4}{x^4-1} = \frac{x^2-4}{(x^2+1)(x-1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}$$

$$= \frac{(Ax+B)(x-1)(x+1) + C(x^2+1)(x+1) + D(x^2+1)(x-1)}{x^4-1}$$

$$= \frac{(A+C+D)x^3 + (B+C-D)x^2 + (-A+C+D)x + (-B+C-D)}{x^4-1}$$

<u>equating coeff^s</u>: ① $A+C+D=0$

② $B+C-D=1$

③ $-A+C+D=0$

④ $-B+C-D=-4$

①&③ $\Rightarrow$ $A=0$ and $C+D=0$ so $C=-D$

② $\Rightarrow$ $B+2C=1$ so "②+④" $4C=-3$ so $C=\frac{-3}{4}$

④ $\Rightarrow$ $-B+2C=-4$

"②-④": $2B=5$ so $B=5/2$

$A=0$ $C=-3/4$

$B=5/2$ $D=3/4$

so $$\frac{x^2-4}{x^4-1} = \frac{5/2}{x^2+1} - \frac{3/4}{x-1} + \frac{3/4}{x+1}$$

50 $$\int \frac{x^2-4}{x^4-1}\,dx = \frac{5}{2}\int \frac{1}{x^2+1}\,dx - \frac{3}{4}\int \frac{1}{x-1}\,dx + \frac{3}{4}\int \frac{1}{x+1}\,dx$$

$$= \frac{5}{2}\arctan(x) - \frac{3}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + C$$