IVT

Continuous functions can't jump.

Fact [Intermediate Value Theorem ("IVT")]: If f is continuous on $[b, c]$ and d is between $f(b)$ and $f(c)$ then there exists e in $[b, c]$ such that $f(e) = d$

Example: Does

$$e^{sin(x\frac{\pi}{2})} = x$$

have any solutions for x?

Try some obvious values:

$$e^{sin 0} = e^0 = 1 > 0$$

$$e^{sin(\pi/2)} = e^1 = e \ = 2.72 > 1$$

$$e^{sin \pi} = e^0 = 1 < 2$$

Let $f(x) = e^{sin(x\frac{\pi}{2})} - x$ so we want to know if there exists x such that $f(x) = 0$.

$$f(0) = 1$$
$$f(1) = e - 1 \approx 1.72$$
$$f(2) = 1 - 2 = -1$$

Key point: f is continuous!

0 is between $f(1)$ and $f(2)$ so by the IVT, there exists e in $[1, 2]$ such that $f(e) = 0$.

Better still, we can use this trick to actually find a solution!

// Demonstrate binary search to find the root at 1.661

$$f(1.6) = 0.1999974573044334$$
$$f(1.661) = 0.00044671501254378576$$
$$f(1.5) = 0.5281149816474726$$
$$f(1.7) = -0.1254169615450087$$
$$f(1.6) = 0.1999974573044334$$
$$f(1.65) = 0.036235559313816834$$
$$f(1.66) = 0.00369563789547267$$
$$f(1.67) = -0.028749143599565707$$
$$f(1.665) = -0.01253926227311819$$
$$f(1.662) = -0.002801246291008308$$
$$f(1.661) = 0.00044671501254378576$$
$$f(1.6615) = -0.001177386453861784$$

Differentiation

Definition: For f a function and b in the domain of f, we say f is differentiable at b if the limit

$$\lim_{x \to b} \frac{f(x) - f(b)}{x - b}$$

exists, and then the derivative of f at b is the value of this limit.

(Saying the limit exists means that there is a *finite* limit.)

f' is the function

$$f'(b) = [\text{ derivative of } f \text{ at } b \text{ }],$$

defined where f is differentiable.

Remark:

$$\lim_{x \to b} \frac{f(x) - f(b)}{x - b} = \lim_{h \to 0} \frac{f(b + h) - f(b)}{h}.$$

Interpretations: For clarity, suppose $f(0) = 0$ and we're considering the derivative at 0

$$f'(0) = \lim_{x \to 0} \frac{f(x) - 0}{x - 0} = \frac{f(x)}{x}.$$

Local linearity: $f'(b) = c$

iff $\frac{f(x)}{x}$ is arbitrarily close to c for all x sufficiently close to but not equal to 0,

iff $f(x)$ is arbitrarily close to cx for all such x (in the sense of a cone rather than a strip),

iff sufficiently close to 0, the graph of $f(x)$ is arbitrarily close to a straight line with slope c.

So generally, a function f is differentiable at b iff the graph of f is "approximately a non-vertical straight line near b", and $f'(b)$ is then the slope of this line. That line is then called the tangent line of f at b.

Rate of change: If f is a function of time, $f'(t)$ is the rate of change of f at time t: if $f(0) = 0$, then for all t sufficiently small, $f(t) = f'(0)t$.

Example: suppose an object moves along a line, and $s(t)$ is its position at time t. By Newton's First Law, if at time b we remove all forces acting on the object, its position after b will be

$$s(t) = s(b) + (t - b)s'(b).$$

(same idea as swinging an object in a circle then releasing it.)

Examples: [sketch of f' for a smooth function f]
$f(x) := x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(2hx+h^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= (\lim_{h\to 0} 2x) + (\lim_{h\to 0} h) \\ &= 2x+0 \\ &= 2x. \end{aligned}$$

Fact: If a function f is differentiable at b, then f is continuous at b.
Proof:

$$\begin{aligned} \lim_{x\to b}(f(x)-f(b)) &= \lim_{x\to b}((x-b)\frac{f(x)-f(b)}{x-b}) \\ &= (\lim_{x\to b}(x-b))(\lim_{x\to b}\frac{f(x)-f(b)}{x-b}) \\ &= 0f'(b) \qquad\qquad = 0 \end{aligned}$$

, so $\lim_{x\to b} f(x) = f(b)$

Examples of non-differentiability: By the fact, if f is not continuous at b, then f is not differentiable at b.

$|x|$ is continuous but not differentiable at 0 (different behaviours on left and right).

$\sqrt{x}$ is continuous but not differentiable at 0 (limit is infinite).

$f(x) = xsin(1/x)$ if $x \neq 0$ and $f(0) = 0$: continuous but not differentiable at 0, since $\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \sin(1/x)$ does not exist.

More notation:

$$\frac{df(x)}{dx} = f'(x).$$

$$\frac{df(t)}{dt} = f'(t).$$

$$\frac{df(x)}{dx} \restriction_b = f'(b).$$

If $y = f(x)$,

$$\frac{dy}{dx} = f'(x).$$

$$\frac{dy}{dx} \restriction_b = f'(b).$$

Also, e.g.

$$\frac{d}{dx}x^2 = \frac{d(x^2)}{dx} = 2x.$$

Higher derivatives: f' is a function, so we can consider $(f')'$, abbreviated to f'' or $f^{(2)}$.

e.g. if $f(x) = x^2$ then $f''(x) = \frac{df'}{dx} = \frac{2x}{dx} = \lim h \to 0 \frac{2(x+h)-2x}{h} = \lim h \to 02 =$ 2.

If $s(t)$ is position of an object at time t, then $s''(t)$ is acceleration at time t (so Newton's 2nd Law in one dimension is $s''(t) = F(t)/m$, where $F(t)$ is the sum of the forces and m is the mass).

$f''' = f^{(3)} = (f'')'$, and so on.

Calculating derivatives

Fact (Linearity of derivatives):

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x).$$

For c a real number,

$$\frac{d}{dx}cf(x) = cf'(x).$$

These follow very easily from definition of derivative.

Product rule: If f and g are differentiable at b, then so is the product $h(x) := f(x)g(x)$, and

$$h'(b) = \frac{d}{dx}f(x)g(x) \restriction_b = f(b)g'(b) + f'(b)g(b).$$

Explanation: for small h,

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &\approx (f(x) + hf'(x))(g(x) + hg'(x)) \\ &= h(f(x)g'(x) + f'(x)g(x)) + h^2 f'(x)g'(x) \\ &\approx h(f(x)g'(x) + f'(x)g(x)). \end{aligned}$$

Differentiating polynomials: Clearly $\frac{d}{dx}1 = 0$ and $\frac{d}{dx}x = 1$.

$\frac{d}{dx}x^2 = \frac{d}{dx}xx = 1x + x1 = 2x$.
$\frac{d}{dx}x^3 = \frac{d}{dx}xx^2 = 1x^2 + x(\frac{d}{dx}x^2) = x^2 + x(2x) = 3x^2$.
$\frac{d}{dx}x^4 = \frac{d}{dx}xx^3 = 1x^3 + x(\frac{d}{dx}x^3) = x^3 + x(3x^2) = 4x^3$.
...
$\frac{d}{dx}x^n = nx^{n-1}$.
So e.g. $\frac{d}{dx}(2x^5 - 3x^2 + 8) = ...$

Quotient rule: We'll see later, as a consequence of the Chain Rule, that

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2}.$$

So by the product rule,

$$\frac{d}{dx}\frac{f(x)}{g(x)} = f(x)\frac{-g'(x)}{g(x)^2} + f'(x)g(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

e.g. rational functions

Differentiating power functions:

$$\frac{d}{dx}b^x = \lim_{h\to 0}\frac{b^{x+h} - b^x}{h} = \lim_{h\to 0}\frac{b^x b^h - b^x}{h} = b^x \lim_{h\to 0}\frac{b^h - 1}{h}.$$

e is the unique real number such that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$, i.e. such that $\frac{d}{dx}e^x = e^x$.

Differentiating trig functions: Fact:

$$\sin' = \cos;$$

$$\cos' = -\sin.$$

[give diagrammatic argument]
By the quotient rule,

$$\tan' = (\sin'\cos - \sin\cos')/\cos^2 = (\cos^2 + \sin^2)/\cos^2 = \sec^2$$