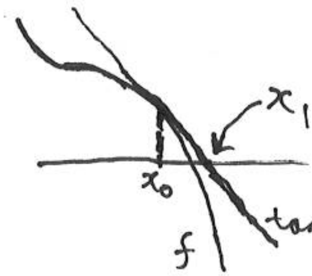
Newton's method

http://ms.mcmaster.ca/~mbays

To find to high accuracy a zero of $f(x)$:

- Let x_0 be close to the zero
- Given x_n, let x_{n+1} be where the tangent line to f at x_n intersects the x-axis

i.e. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$



tangent line:

$y = f(x_0) + f'(x_0)(x - x_0)$

$y = 0 \iff x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

so set $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

- If x_0 was close enough to the zero and f is well-behaved near it (twice differentiable)

then x_n rapidly converges to the zero.

Continuation;
- poly^ns
- Quotient rule
- b^x, e
- sin', tan'

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example:

$$e^{\sin(\frac{\pi}{2}x)} = x$$

Let $f(x) = e^{\sin(\frac{\pi}{2}x)} - x$

$$f'(x) = \frac{\pi}{2}\cos(\frac{\pi}{2}x)e^{\sin(\frac{\pi}{2}x)} - 1$$

$$\begin{aligned} x_0 &= 2 \\ x_1 &= 1.6110154703516575 \\ x_2 &= 1.6609035259641185 \\ x_3 &= 1.6611375113211584 \\ x_4 &= 1.6611375194231848 \\ x_5 &= 1.6611375194231848 \end{aligned}$$

but

$$\begin{aligned} x_0 = 3, x_1 &= 0.36787944117144233 \\ x_2 &= -0.7004442394773245 \\ x_3 &= 0.8682758685413106 \\ x_4 &= 13.55349391880111 \\ \ldots x_{100} &= -201.15250823446343 \\ \ldots x_{130} &= 1.6611375194231848 \end{aligned}$$

Failure:

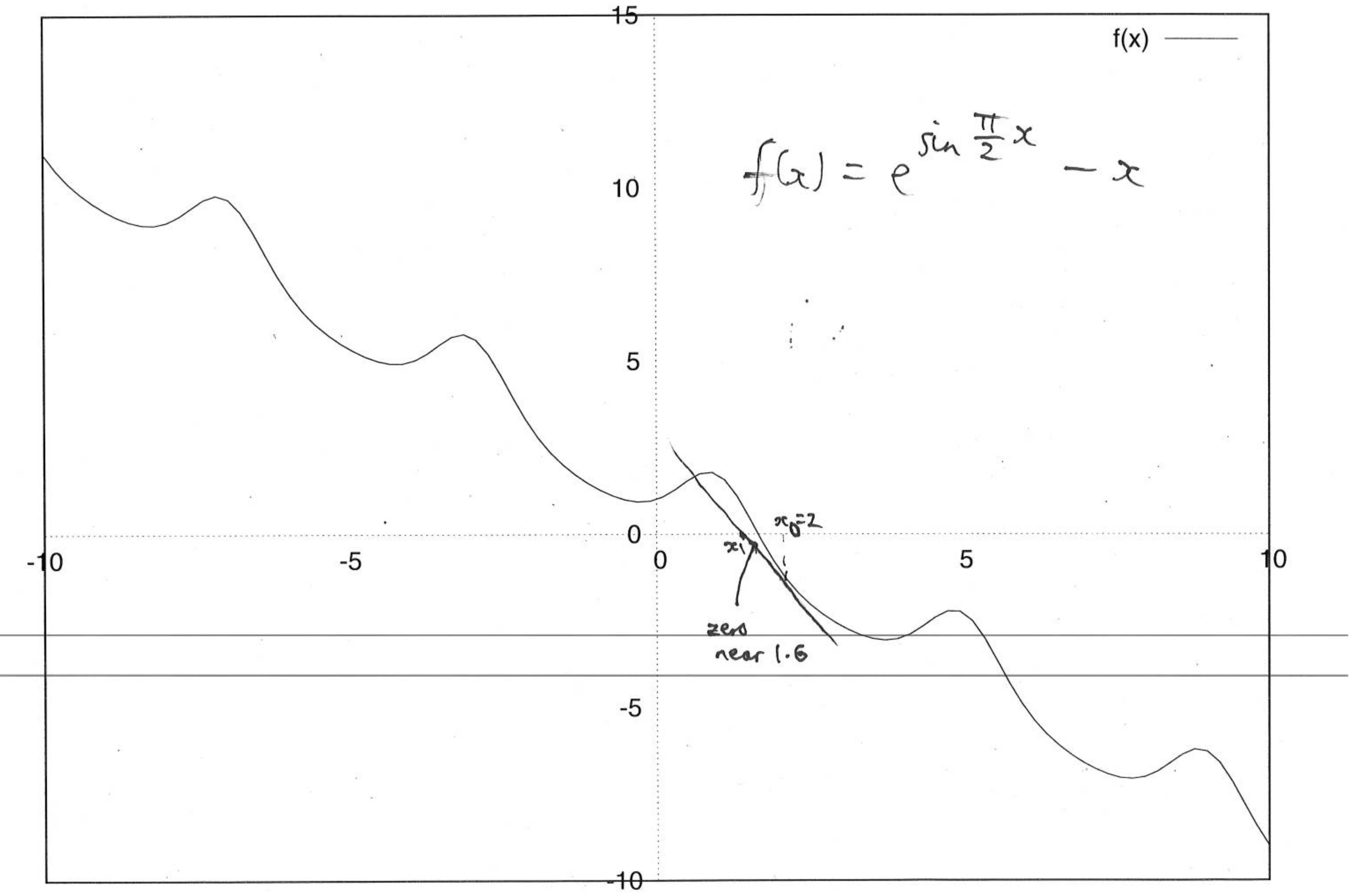
$$f(x) := x^{1/3}$$
$$f'(x) = (1/3)x^{-2/3}$$
$$\frac{f(x)}{f'(x)} = 3x$$

So $x_{n+1} = -2x_n$, so diverges unless $x_0 = 0$.

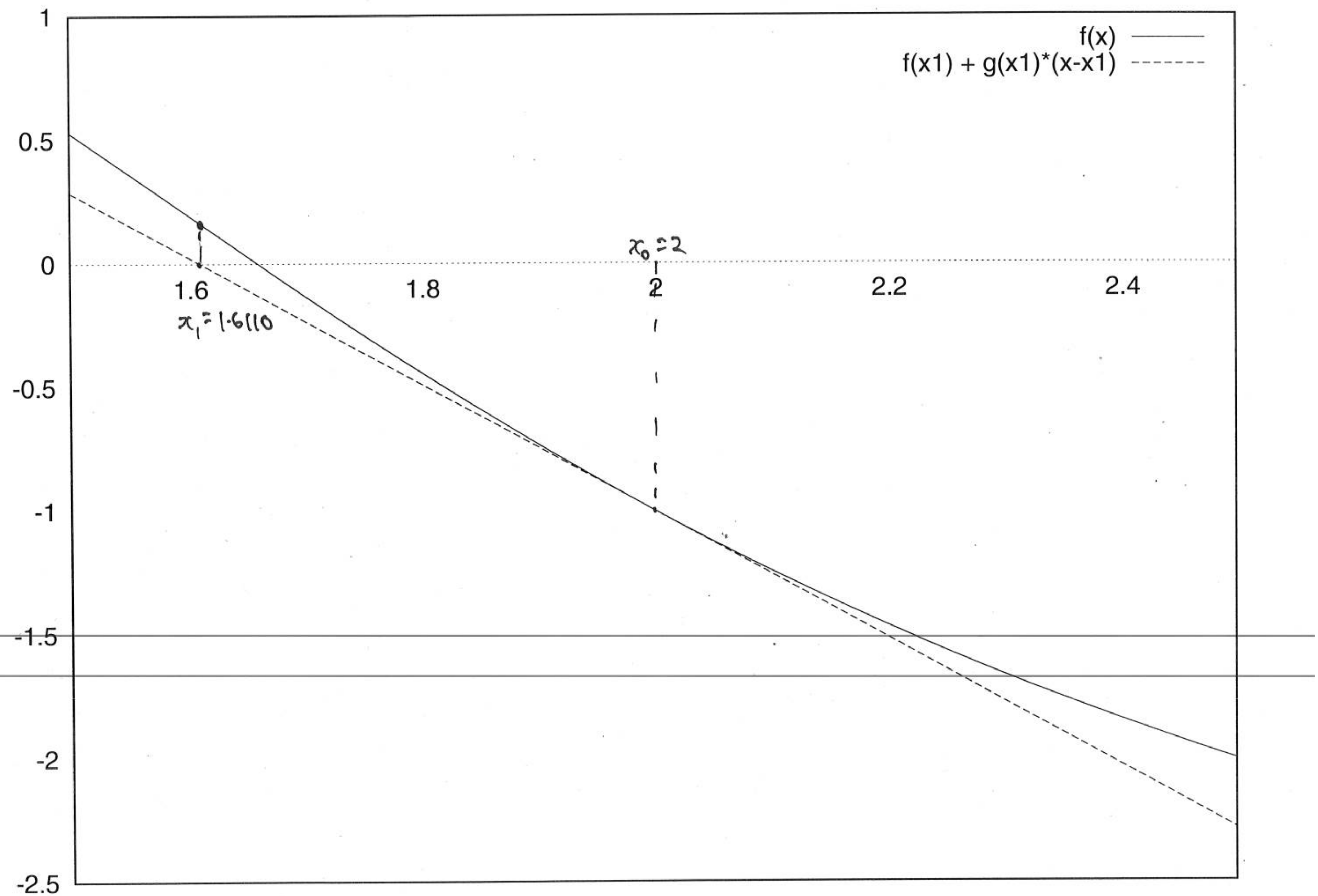
Chain rule

$$\boxed{(fog)' = g'(f'og)}$$

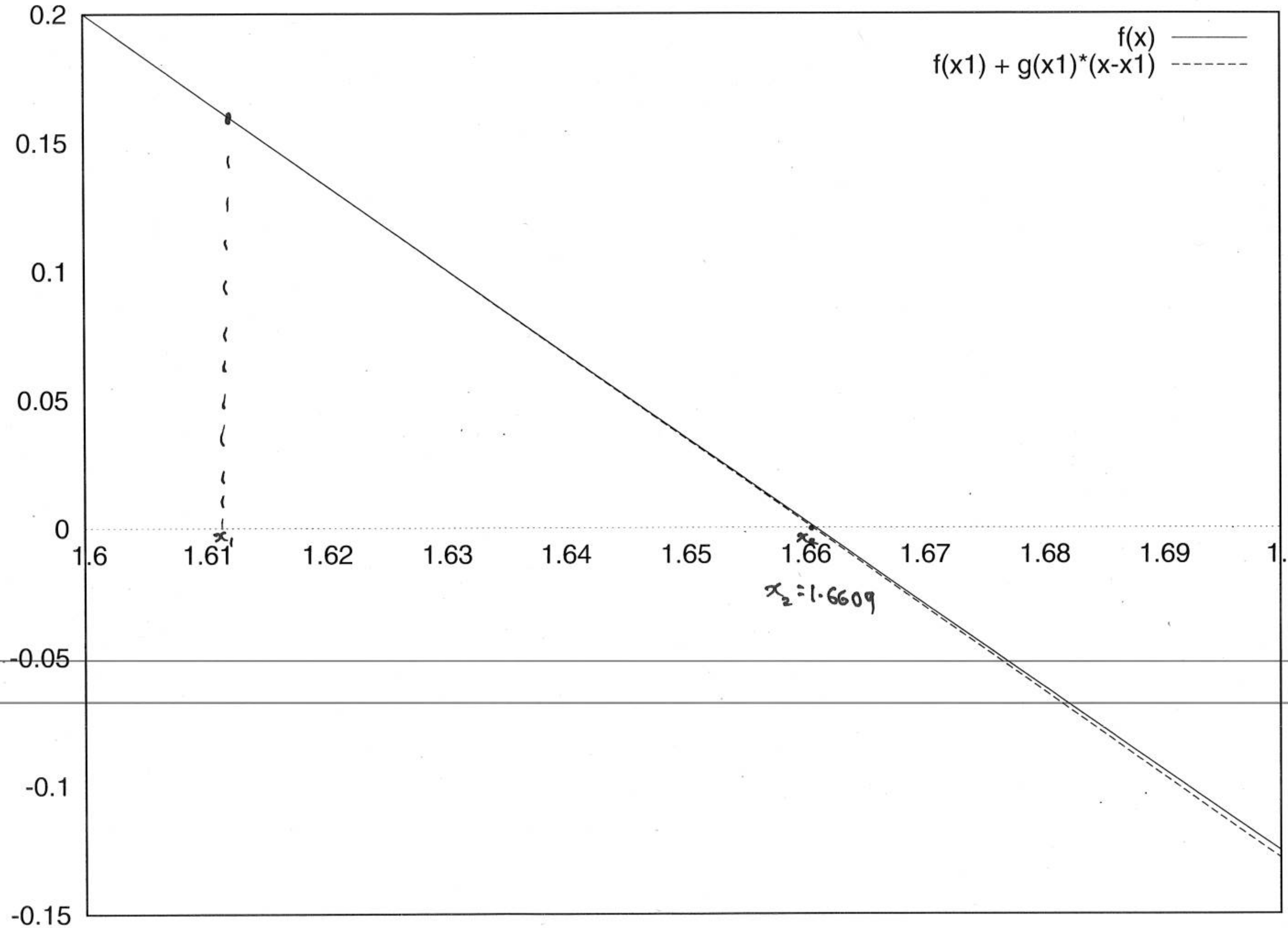
i.e. if $h(x) = f(g(x))$
then $h'(x) = g'(x) \cdot (f'(g(x))$

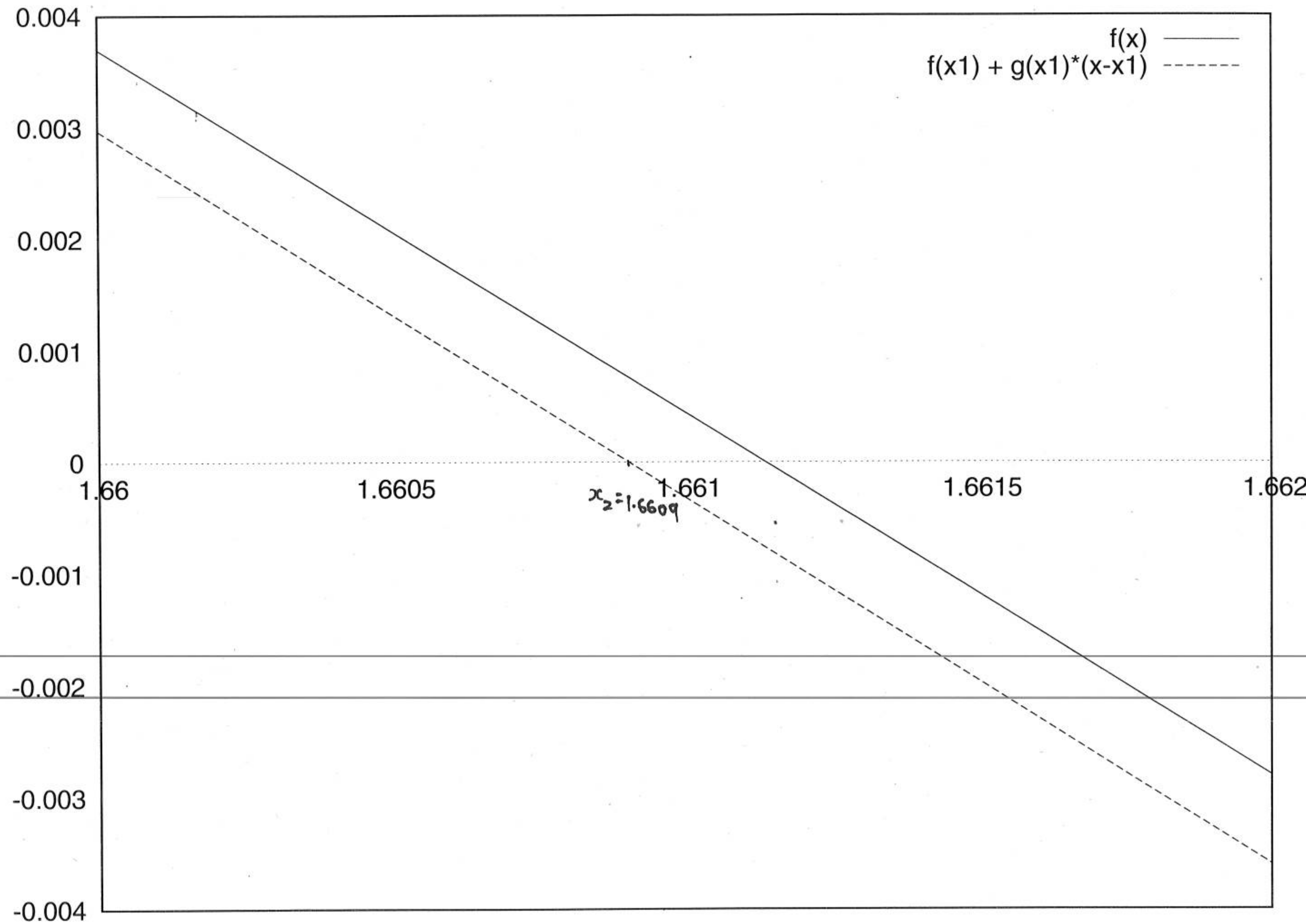


f(x)
$f(x) = e^{\sin \frac{\pi}{2} x} - x$
$x_0 = 2$
x_1
zero
near 1.6
15
10
5
0
-5
-10
-10
-5
0
5
10



f(x)
f(x1) + g(x1)*(x-x1)
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
1.6
1.8
2
2.2
2.4
$x_0 = 2$
$x_1 = 1.6110$


f(x)
f(x1) + g(x1)*(x-x1)
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
1.6
1.61
1.62
1.63
1.64
1.65
1.66
1.67
1.68
1.69
1.7
x1
x2
x2=1.6609


f(x)
f(x1) + g(x1)*(x-x1)
0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
-0.004
1.66
1.6605
1.661
1.6615
1.662
$x_2 = 1.6609$