Adding up

Summation notation: Given numbers a_i and integers $m \leq n$,

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n.$$

So e.g. if we have 50 boxes containing balls, and the i^{th} box contains B_i balls, then the total number of balls in the boxes is

$$\sum_{i=1}^{50} B_i.$$

Similarly, given a function f and integers $m \leq n$,

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + \ldots + f(n).$$

So e.g. we can abbreviate the sum

$$1 + 2 + \ldots + n$$

as

$$\sum_{i=1}^{n} i.$$

Remarks: By basic algebra,

$$\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$$

e.g. $ca_1 + ca_2 + ca_3 = c(a_1 + a_2 + a_3)$

$$\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right).$$

$$\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} (a_i + (-b_i))$$
$$= \left(\sum_{i=m}^{n} a_i\right) + \sum_{i=m}^{m} (-b_i)$$
$$= \left(\sum_{i=m}^{n} a_i\right) - \left(\sum_{i=m}^{n} b_i\right)$$

Also, if $m \leq n < s$, then

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{s} a_i = \sum_{i=m}^{s} a_i$$

We don't have to use i as the index; e.g.

$$\sum_{n=s}^{t} f(n) = \sum_{i=s}^{t} f(i)$$

Examples:

$$\sum_{i=n}^{n} a_i = a_n$$

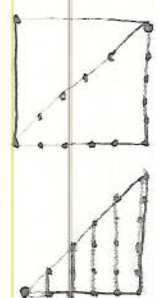
$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=0}^{n} 1 = n+1$$

$$\sum_{i=n}^{n} 1 = 1$$

$$\sum_{i=m}^{n} 1 = n+1-m$$

So $T_n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$



n^2 points in square $= 2 \cdot T_n - n$

points on diagonal

number of points is T_6

The n^{th} /triangular number/ is

$$T_n := \sum_{i=0}^{n} i. \quad = 0+1+2+\ldots+n$$

$$\begin{aligned} 2T_n &= \sum_{i=0}^{n} i + \sum_{i=0}^{n} (n-i) \\ &= \sum_{i=0}^{n} n \\ &= (n+1)\,n. \end{aligned}$$

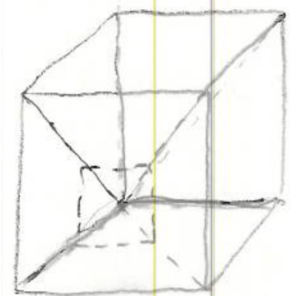
$n+(n-1)+(n-2)+\ldots+(n-n)$

So

$$T_n = \frac{n\,(n+1)}{2}.$$

Note that we can then easily calculate e.g.

$$\begin{aligned} \sum_{i=37}^{1337} i &= \sum_{i=0}^{1337} i - \sum_{i=0}^{36} i \\ &= T_{1337} - T_{36} \\ &= \frac{(1337)\,(1338) - (36)\,(37)}{2} \\ &= 893787. \end{aligned}$$



Cube = 3 Cone
− 3 triangle
+ line

so $n^3 = 3S_n - 3T_n + n$

$3S_n = n - n^3 - \frac{3n(n+1)}{2}$

Sum of consequetive squares:

$$S_n := \sum_{i=0}^{n} i^2$$

$$\boxed{S_n = \frac{n(n+1)(2n+1)}{6}}$$

We can test this formula: clearly it works for $n = 0$, and if it works for $n = k - 1$ then

$$\begin{aligned} S_k &= S_{k-1} + k^2 \\ &= \frac{(k-1)(k)(2k-1)}{6} + k^2 \\ &= \frac{k((k-1)(2k-1)+6k)}{6} \\ &\cancel{= \frac{k((k-1)(2k-1)+6k)}{6}} \\ &= \frac{k(k+1)(2k+1)}{6}. \end{aligned}$$

(since $((k-1)+2)((2k-1)+2) = (k-1)(2k-1)+(4k-2)+(2k-2)+4 = (k-1)(2k-1)+6k$)

So the formula works for all n.

Estimating areas

Areas of shapes defined by straight lines (rectangles, triangles, polygons etc) are easy to calculate. But what about when the boundary is a curve?

e.g. What is the area of an ellipse? What is the area below a catenary?

Area beneath a graph: Let $[a, b]$ be an interval and let $f(x)$ be a function continuous and non-negative on the interval. We will try to estimate the area bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.

e.g. $f(x) = x^2$, $[a, b] = [0, 1]$.