So the sum of the areas is

$$A_n = \sum_{i=1}^{n} RectArea_i$$
$$= \sum_{i=1}^{n} \frac{i^2}{n^3} \qquad\qquad = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

e.g. with $n = 10$: $A_{10} = 10{*}11{*}21/6000 = 0.385$. with $n = 1000$: $A_{1000} = (1000 * 1001 * 2001) / (6 * 1000 * 1$
0.3338335

Now: since the estimate gets more and more accurate for larger n, we can expect that the area *is* the limit $\lim_{n\to\infty} A_n = \frac{1}{3}$.

Remarks: It wasn't important to our reasoning that we took the value of f at the right end-point of each interval to define the height of the corresponding rectangle. Taking the value of f at *any* point of the interval should work just as well.

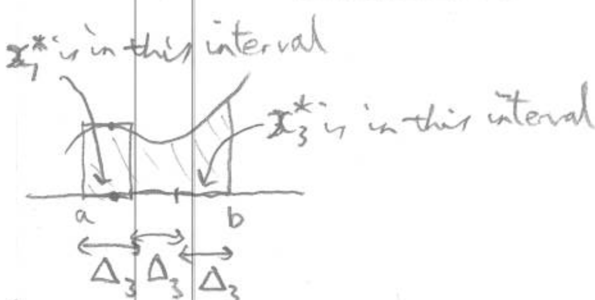
Sometimes, we won't be able to find a nice formula for the limit as $n \to \infty$ as we could above. Still, we expect the above approach to give a good estimate (assuming f is "reasonable").

Definite Integrals

Definition: A function f is integrable on an interval $[a, b]$ if the limit $\lim_{n\to\infty} S_n$ of Riemann sums exists and is the same for any choice of Riemann sums, and in this case that limit is the definite integral of f from a to b.

Here, a Riemann sum S_n is the sum

$$S_n = \sum_{i=1}^{n} \Delta_n f(x_i^*)$$



sample point

where $\Delta_n = \frac{b-a}{n}$, and x_i^* is a choice of a point in the interval

$$[a + (i-1)\Delta_n, a + i\Delta_n].$$

So the definite integral is the limit of Riemann sums; but if f is ill-behaved, this limit might depend on exactly how we calculate the Riemann sums (what points we calculate f at), so then we don't get a well-defined integral and we say that f is not integrable on $[a, b]$. Luckily...

Theorem: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

$\left(\sum_{i=n}^{m} f(i)\right)$

Notation: We write

$$\boxed{\int_a^b f(x)\,dx}$$

for the definite integral from a to b of f.

"dx" here should be read as notation indicating the variable we are integrating with respect to, much like the $\frac{d}{dx}$ of differentiation. So e.g.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$\frac{d}{dx} f(x) = f'(x)$ $\qquad$ $\frac{d}{dy} f(y) = f'(y)$

So

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{(b-a)}{n}\sum_{i=1}^{n} f(x_i^*)\right)$$

where for each n, each x_i^* is a choice of point in the n^{th} interval, and the limit exists and doesn't depend on these choices (which is true if f is continuous on $[a,b]$). i^{th}

So e.g. we saw above that

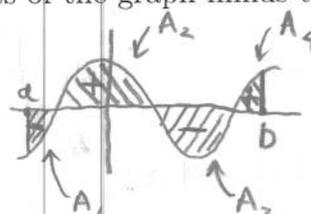
$$\int_0^1 x^2 dx = \frac{1}{3}$$

If f is non-negative on $[a,b]$, then $\int_a^b f(x)\,dx$ is precisely the limit of the estimates to the area beneath the graph we discussed above. We *define* that area to be the integral. More generally:

Interpretation/Definition: If $a \le b$, the signed area (or net area) between the graph of f, the x-axis, and the vertical lines $y=a$ and $y=b$ is defined to be $\int_a^b f(x)\,dx$.

So the signed area is the sum of the areas below the positive parts of the graph minus the sum of the areas above the negative parts.

$x(x+1)(x-1)$



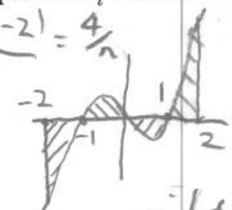
$\int_a^b f(x)dx = A_2 + A_4 - A_1 - A_3$

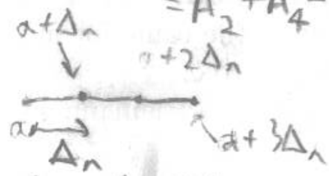
Example:

$$\int_{-2}^{2}(x^3 - x)\,dx$$

We can use right-hand endpoints, i.e. choosing sample point x_i^* to be $-2 + i\Delta_n$

$\Delta_n = \frac{2-(-2)}{n} = 4/n$





$i\Delta_n$

$$\begin{aligned}
\int_{-2}^{2}(x^3 - x)\,dx &= \lim_{n\to\infty}\Delta_n\sum_{i=1}^{n} f(x_i^*)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n} f\left(-2+\frac{4i}{n}\right)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-2+\frac{4i}{n}\right)^3 - \left(-2+\frac{4i}{n}\right)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-6+\frac{(3)(4)(4i)-4i}{n}+\frac{(3)(-2)(4i)^2}{n^2}+\frac{(4i)^3}{n^3}\right)\\
&= \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^{n}\left(-6+44\frac{i}{n}-96\frac{i^2}{n^2}+64\frac{i^3}{n^3}\right)\\
&= \lim_{n\to\infty}4\left(-6+44\frac{n(n+1)}{2n^2}-96\frac{n(n+1)(2n+1)}{6n^3}+64\frac{(n(n+1))^2}{4n^4}\right)\\
&= 4\left(-6+\frac{44}{2}-96\frac{1}{3}+64\frac{1}{4}\right)\\
&= 4(-6+22-32+16)\\
&= 0
\end{aligned}$$

right-hand endpoint of i^{th} interval

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} -6 = -6n$

$\frac{44}{n^2}\sum_{i=1}^{n} i$

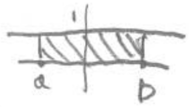
(we used here the formula

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

see Appendix E problem 40 for a rather nice proof.)

Facts:

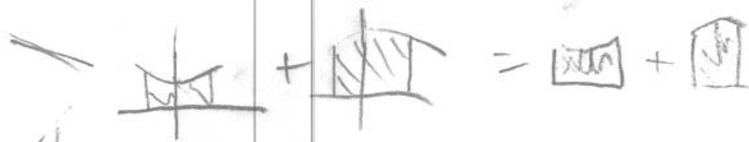
(i) $\int_a^b 1 dx = b - a$



(ii) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

e.g. $\int_0^1 5x^2\,dx = 5\int_0^1 x^2\,dx = 5/3$

(iii) $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$



(iv) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(v) $\int_a^a f(x)\,dx = 0$

Remark: It follows from (iv) and (v) that $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ so in terms of the signed area interpretation, taking the endpoints the "wrong way round" introduces a minus sign.