Lecture 24 Indefinite integrals and the net change theorem

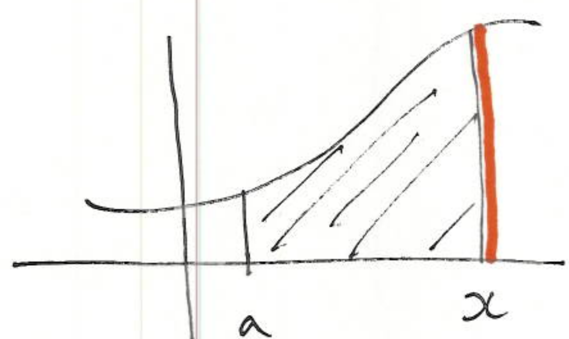
(Dr Bays' class)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \quad , \text{ where } \Delta x = \frac{b-a}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_n f(x_i^*) \quad , \text{ where } \Delta_n = \frac{b-a}{n}$$

FTC I Let $g(x) = \int_a^x f(t)\,dt$ then

$$g'(x) = f(x).$$

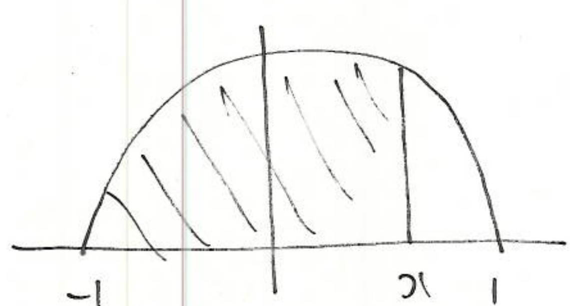


Consider

$$h(x) = \int_a^{\sin(x)} f(t)\,dt$$

Ex. $f(t) = \sqrt{1-t^2}$

$$g(x) = \int_{-1}^{x} \sqrt{1-t^2}\,dt$$



$g'(x) = \sqrt{1-x^2}$ by FTC.

$$h(x) = \int_{-1}^{\sin(x)} \sqrt{1-t^2}\,dt$$

Notice that $h(x) = g(\sin(x))$

Chain rule to differentiate

$$h'(x) = g'(\sin(x))\cos(x)$$
$$= \sqrt{1-(\sin(x))^2}\,\cos(x).$$

<u>Notation</u>

$$\int f(x)\,dx = g(x) + C, \text{ where}$$

$$g(x) = \int_a^x f(t)\,dt \quad \underline{\text{or}} \text{ any}$$

other antiderivative of f.

<u>definite</u> integral $\int_a^b f(t)\,dt$ – Riemann sum
a <u>number</u>

<u>indefinite</u> integral $\int f(x)\,dx = F(x) + C,$
where $F'(x) = f(x)$
– is a <u>family of functions</u>

FTC II

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

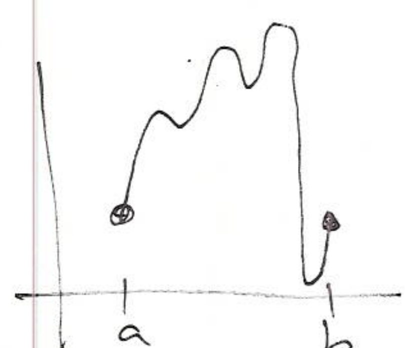
$$= F(b) - F(a), \quad \text{where}$$

$$F'(x) = f(x).$$

Equivalently:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

integral of the rate of change of F = **net** change in F



Ex. A particle moving in a straight line has velocity $v(t) = t^3 - 9t^2 + 18t$. Find the displacement and the distance travelled as t goes from 0 to 6.

$v(t) = s'(t)$ ~~st~~ where $s(t)$ = position at time t.

displacement = net change in position

$= s(6) - s(0)$

$$= \int_0^6 s'(t)\,dt$$

$$= \int_0^6 v(t)\,dt.$$

$$\int_0^6 v(t)\,dt = \int_0^6 (t^3 - 9t^2 + 18t)\,dt$$

$$= \int_0^6 t^3\,dt - 9\int_0^6 t^2\,dt + 18\int_0^6 t\,dt$$

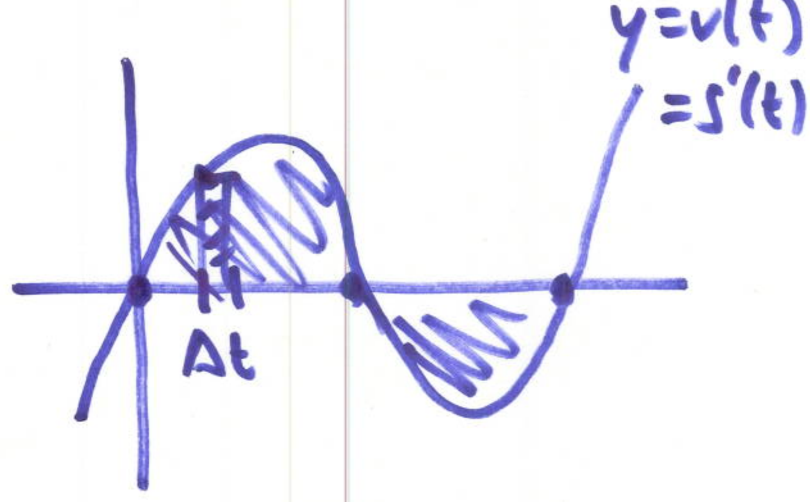
$$= \left[\frac{1}{4}t^4\right]_0^6 - 9\left[\frac{1}{3}t^3\right]_0^6 + 18\left[\frac{1}{2}t^2\right]_0^6$$

$$= \frac{1}{4}6^4 - \frac{1}{4}0^4 - 9\left(\frac{1}{3}6^3 - \frac{1}{3}0^3\right) + 18\left(\frac{1}{2}6^2 - \frac{1}{2}0^2\right)$$

$$= \frac{1}{4}36\cdot 36 - 3\cdot 6\cdot 36 + 9\cdot 36$$

$$= 0.$$

$$v(t) = t^3 - 9t^2 + 18t$$
$$= t(t^2 - 9t + 18)$$
$$= t(t-3)(t-6)$$



$s(0) = 0$

distance = velocity × time

distance travelled = area enclosed by the velocity-time curve

$$= \int_0^3 v(t)\,dt - \int_3^6 v(t)\,dt$$

$$= \Big[s(t)\Big]_0^3 - \Big[s(t)\Big]_3^6$$

$$= s(3) - s(0) - \big(s(6) - s(3)\big)$$

$$= 2s(3) - s(0) - s(6)$$

$$s(t) = \int s'(t)\,dt$$

$$= \int (t^3 - 9t^2 + 18t)\,dt$$

$$= \frac{1}{4}t^4 - 3t^3 + 9t^2 + C.$$

dist. travelled $= 2s(3) - s(0) - s(6)$

$$= 81/2.$$