Fundamental Theorem of Calculus (FTC) Reiterated

Theorem [FTC]:

(I) Let f be continuous on $[a, b]$. Then

$$F(x) = \int_a^x f(t)\, dt$$

is differentiable on (a, b) and continuous on $[a, b]$, and for x in (a, b) we have

$$F'(x) = f(x).$$

So for continuous f,

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x),$$

or in the indefinite integral notation:

$$\frac{d}{dx} \int f(x)\, dx = f(x),$$

(II) If f is continuous on $[a, b]$ and $F' = f$ on $[a, b]$, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

In other words: if g is differentiable on $[a, b]$ with continuous derivative, then

$$\int_a^b g'(t)\, dt = [g(t)]_a^b = g(b) - g(a).$$

Idea of proof:

(I)

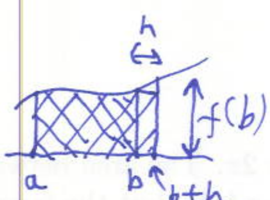
replace b with x

$$
\begin{aligned}
F'(b) &= \lim_{h\to 0} \frac{F(b+h) - F(b)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{b+h} f(t)\, dt - \int_a^b f(t)\, dt\right) \\
&= \lim_{h\to 0} \frac{1}{h}\int_b^{b+h} f(t)\, dt \\
&= \lim_{h\to 0} \frac{1}{h}[\text{signed area between t=b and t=b+h}]
\end{aligned}
$$

$\approx hf(b)$ (rectangle width h height f(b))

(since f is continuous at b and integrable around b)

$$= f(b)$$



(II) By (I), $\int_a^x g'(t)\, dt$ is an antiderivative of $g'(t)$ on (a, b). So by the MVT, $g(x) = \int_a^x g'(t)\, dt + C$ on (a, b), and hence by continuity on $[a, b]$.

Since $\int_a^a g'(t)\, dt = 0$, we must have $C = g(a)$. So

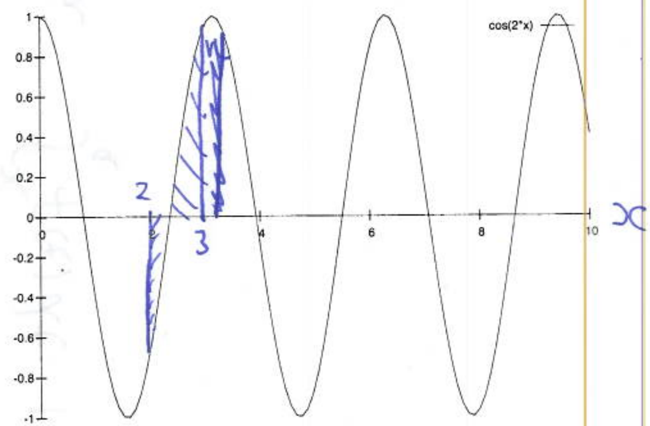
$$\int_a^b g'(t)\, dt = g(b) - g(a).$$

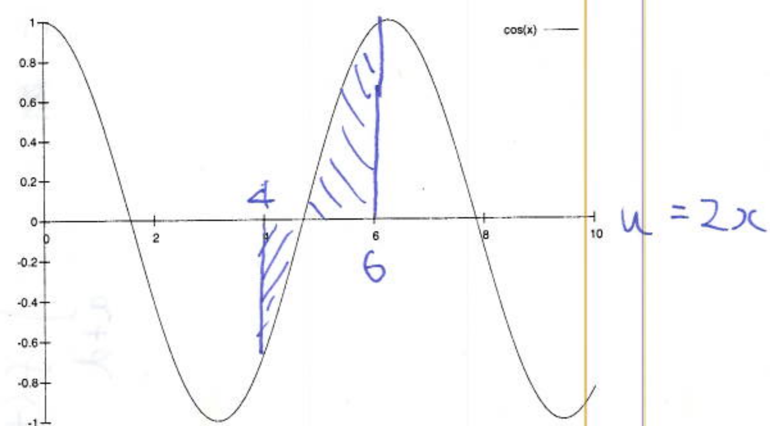
Substitution

Examples and intuition

We know $\int_2^3 \cos(x)\,dx = \sin(3) - \sin(2)$ $(= -0.768)$.

Consider $\int_2^3 \cos(2x)$.





Let $u = 2x$. The area between $x = 2$ and $x = 3$ for $\cos(2x)$ corresponds to the area between $u = 4$ and $u = 6$ for $\cos(u)$ - but the former is squashed by a constant factor of 2 relative to the latter.

We can compensate for the squashing by multiplying $\cos(2x)$ by a constant factor of 2, so we expect:

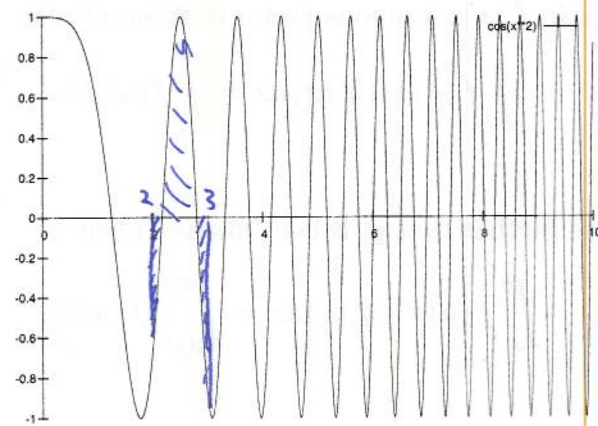
$$\int_2^3 2\cos(2x)\,dx = \int_4^6 \cos(u)\,du = \sin(6) - \sin(4)\,.$$

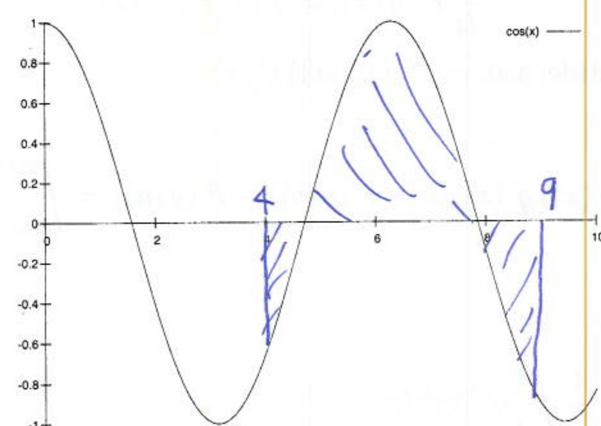
Indeed, by the chain rule, $\sin(2x)$ is an antiderivative of $2\cos(2x)$, so this is right.

We deduce

$$\int_2^3 \cos(2x)\,dx = \frac{1}{2}\int_2^3 2\cos(2x)\,dx = \frac{\sin(6) - \sin(4)}{2} = 0.239$$

Now consider $\int_2^3 \cos(x^2)\, dx$





Let $u = x^2$. The area between $x = 2$ and $x = 3$ for $\cos(x^2)$ corresponds to the area between $u = 4$ and $u = 9$ for $\cos(u)$ - but the former is squashed by a factor of $\frac{d}{dx}x^2 = 2x$ relative to the latter.

We can compensate for the squashing by multiplying $\cos(x^2)$ by a factor of $2x$, so we expect:

$$\int_2^3 2x \cos\left(x^2\right) dx = \int_4^9 \cos(u)\, du = \sin(9) - \sin(4).$$

Again, we can confirm this using the chain rule: $\sin(x^2)$ is an antiderivative of $2x\cos(x^2)$.

Note that we have actually discovered **nothing** about $\int_2^3 \cos(x^2)$! Instead, we have found an entirely **different** integral, namely

$$\int_2^3 2x \cos\left(x^2\right) dx.$$

There is no way to get from that to any information about $\int_2^3 \cos(x^2)$!

Formal formulation

Theorem [substitution rule]:

(a) For indefinite integrals: Suppose f is continuous and g is differentiable. Then

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$.

(b) For definite integrals: Suppose further that g' is continuous. Then

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof:

(a) If F is an antiderivative of f, then by the chain rule

$$\frac{d}{dx}F(g(x)) = f(g(x))\,g'(x)$$

so $F(u) = F(g(x))$ is an antiderivative of $f(g(x))\,g'(x)$.

(b) Now by FTC-II

$$\int_a^b f(g(x))\,g'(x)\,dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du$$

Further examples

$$\begin{aligned}
\int_{-1}^{2} x^3 e^{x^4}\,dx &= \frac{1}{3}\int_{-1}^{2} e^{x^4}\,3x^3\,dx \\
&= \frac{1}{3}\int_{(-1)^4}^{2^4} e^u\,du \qquad \left(u = x^4, \frac{du}{dx} = 4x^3\right) \\
&= \left[\frac{1}{3}e^u\right]_{(-1)^4}^{2^4} \\
&= \left[\frac{1}{3}e^u\right]_{1}^{16} \\
&= \frac{e^{16} - e}{3} \\
&= 2.96 * 10^6
\end{aligned}$$

$$\begin{aligned}
\int x^3 e^{x^4}\,dx &= \frac{1}{3}\int e^{x^4}\,3x^3\,dx \\
&= \frac{1}{3}\int e^u\,du \qquad \left(u = x^4, \frac{du}{dx} = 4x^3\right) \\
&= \frac{1}{3}e^u + C \\
&= \frac{e^{x^4}}{3} + C
\end{aligned}$$