Formal formulation

Theorem [substitution rule]:

(a) For indefinite integrals: Suppose f is continuous and g is differentiable. Then

$$\int f(g(x))\, \underbrace{g'(x)\, dx}_{} = \int f(u)\, du$$

where $u = g(x)$. $\quad g'(x) = \frac{du}{dx}$

(b) For definite integrals: Suppose further that g' is continuous. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof:

(a) If F is an antiderivative of f, then by the chain rule

$$\frac{d}{dx} F(g(x)) = f(g(x))\, g'(x)$$

so $F(u) = F(g(x))$ is an antiderivative of $f(g(x))\, g'(x)$.

(b) Now by FTC-II

$$\int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\, du$$

$f(t) = e^t \quad g(t) = t^4$

Further examples

$e^{x^4} = f(g(x))$

$g'(x) = 4x^3$

$$\begin{aligned}
\int_{-1}^{2} x^3 e^{x^4}\, dx &= \frac{1}{\cancel{3}4} \int_{-1}^{2} e^{x^4}\, \cancel{3}4 x^3\, dx \\
&= \frac{1}{\cancel{3}4} \int_{(-1)^4}^{2^4} e^u\, du \\
&= \frac{1}{4}\left[\cancel{\frac{1}{3}} e^u\right]_{(-1)^4}^{2^4} \\
&= \frac{1}{4}\left[\cancel{\frac{1}{3}} e^u\right]_{1}^{16} \\
&= \frac{e^{16} - e}{\cancel{3}4} \\
&= \cancel{2.96 \times 10^6}
\end{aligned}$$

$$\left(u = x^4, \frac{du}{dx} = 4x^3\right)$$

$$\begin{aligned}
\int x^3 e^{x^4}\, dx &= \frac{1}{\cancel{3}4} \int e^{x^4}\, \cancel{3}4 x^3\, dx \\
&= \frac{1}{\cancel{3}4} \int e^u\, du \\
&= \frac{1}{\cancel{3}4} e^u + C \\
&= \frac{e^{x^4}}{\cancel{3}4} + C
\end{aligned}$$

$$\left(u = x^4, \frac{du}{dx} = 4x^3\right)$$

$f(u) = \frac{1}{\sqrt{u}}$

$g(x) = u = 2+3x$

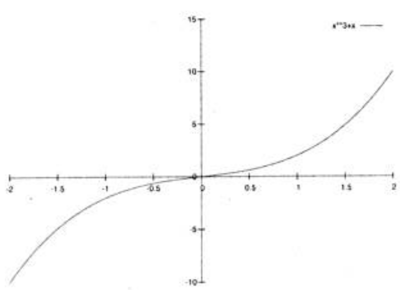
$$\begin{aligned}
\int \frac{1}{\sqrt{2+3x}}dx &= \frac{1}{3}\int \frac{1}{\sqrt{2+3x}}3dx \\
&= \frac{1}{3}\int \frac{1}{\sqrt{u}}du \qquad \left(u = 2+3x, \frac{du}{dx} = 3\right) \\
&= \frac{1}{3}2\sqrt{u} + C \\
&= \frac{2}{3}\sqrt{2+3x} + C
\end{aligned}$$

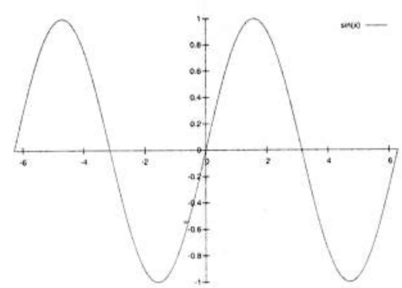
$$\begin{aligned}
\int_1^e \frac{2x + \ln x}{x}dx &= \int_1^e \frac{2x}{x}dx + \int_1^e \frac{\ln x}{x}dx \\
&= \int_1^e 2dx + \int_0^1 udu \qquad \left(u = \ln x, \frac{du}{dx} = \frac{1}{x}\right) \\
&= 2[x]_{x=1}^{x=e} + \left[\frac{u^2}{2}\right]_{u=0}^{u=1} \\
&= 2(e-1) + \frac{1}{2} - 0 \\
&= 2e - \frac{3}{2}
\end{aligned}$$

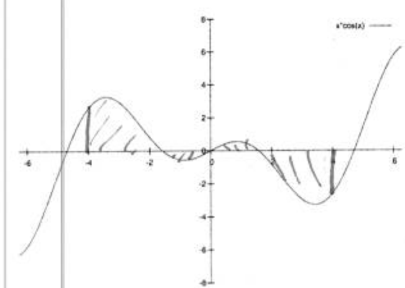
$\int_0^1 u\,du = \int_0^1 x\,dx = \int_0^1 t\,dt = \int_0^1 y\,dy$

Exploiting symmetry

Suppose $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$



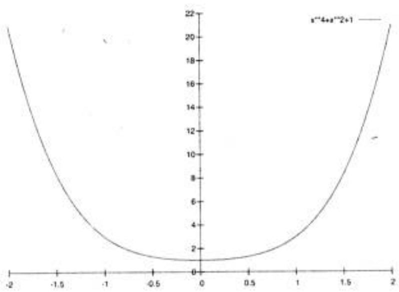


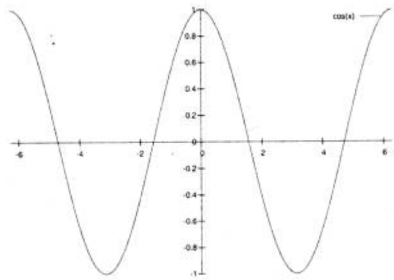


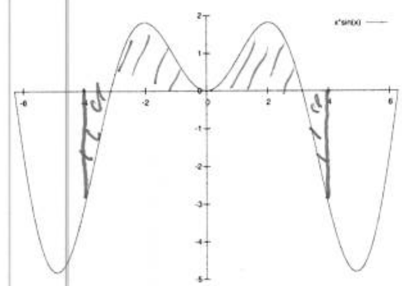
Then

$$\begin{aligned}
\int_{-a}^a f(x)\,dx &= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \\
&= \int_{-a}^0 -f(-x)\,dx + \int_0^a f(x)\,dx \\
&= \int_a^0 f(u)\,du + \int_0^a f(x)\,dx \qquad \left(u = -x, \frac{du}{dx} = -1\right) \\
&= -\int_0^a f(u)\,du + \int_0^a f(x)\,dx \\
&= 0
\end{aligned}$$

Similarly, if $f(x)$ is an even function, i.e. $f(-x) = f(x)$







Then

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= \int_{-a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad \left(u = -x, \frac{du}{dx} = -1\right) \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

$$\cancel{= 2\int_{0}^{a} f(x)\,dx}$$