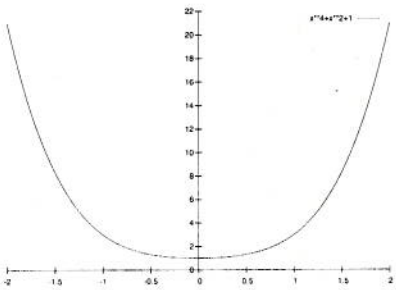
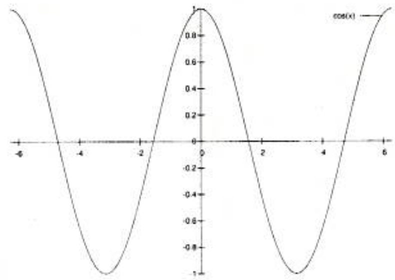
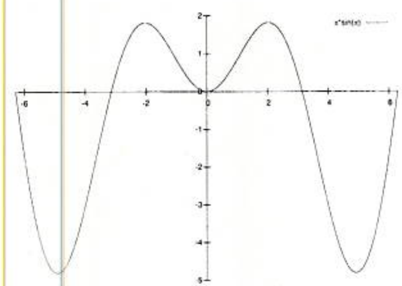
6.1 is not on Midterm 2

Then

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= \int_{-a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \\
&= -\int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad \left(u = -x, \frac{du}{dx} = -1\right) \\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx \\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

$ln|x|$ as an antiderivative of $\frac{1}{x}$

Recall that $\frac{d}{dx}lnx = \frac{1}{x}$. So e.g. it does follow that $\int_1^2 \frac{1}{x}dx = ln2 - ln1$.
But lnx is only defined for $x > 0$, while $\frac{1}{x}$ is also defined for $x < 0$.

Cunning trick: When $x > 0$:

$$\frac{d}{dx}ln|x| = \frac{d}{dx}lnx = \frac{1}{x}$$

When $x < 0$:

$$\frac{d}{dx}ln|x| = \frac{d}{dx}ln(-x) = -\frac{1}{-x} = \frac{1}{x}$$

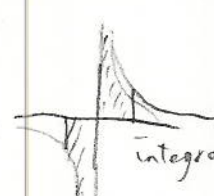
So we can write

$$\int \frac{1}{x}dx = \ln|x| + C,$$

(meaning that this is family of all antiderivatives when we restrict to an interval not containing 0)

Warning: $\frac{1}{x}$ is **not** integrable on any interval containing 0. So e.g.

$$\int_{-1}^{1} \frac{1}{x}dx$$



integral undefined

does not exist (and in particular is **not** equal to $\cancel{\ln|1| + \ln|-1|} = 0$, even though the function is odd!).

$\ln|1| - \ln|-1| = 0$

$$\int_{-2}^{-1} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-2}^{-1} = \ln|-1| - \ln|-2| = \ln 1 - \ln 2 \quad \checkmark$$

Example:

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx \quad = \int \frac{1}{\cos(x)} \sin(x)\,dx$$

$$= -\int \frac{1}{u}\frac{du}{dx}dx \qquad \left(u = \cos(x), \frac{du}{dx} = -\sin(x)\right)$$

$$= -\int \frac{1}{u}du$$

$$= -\ln|u| + C \qquad \cancel{= -\ln|\cos(x)| + C}$$

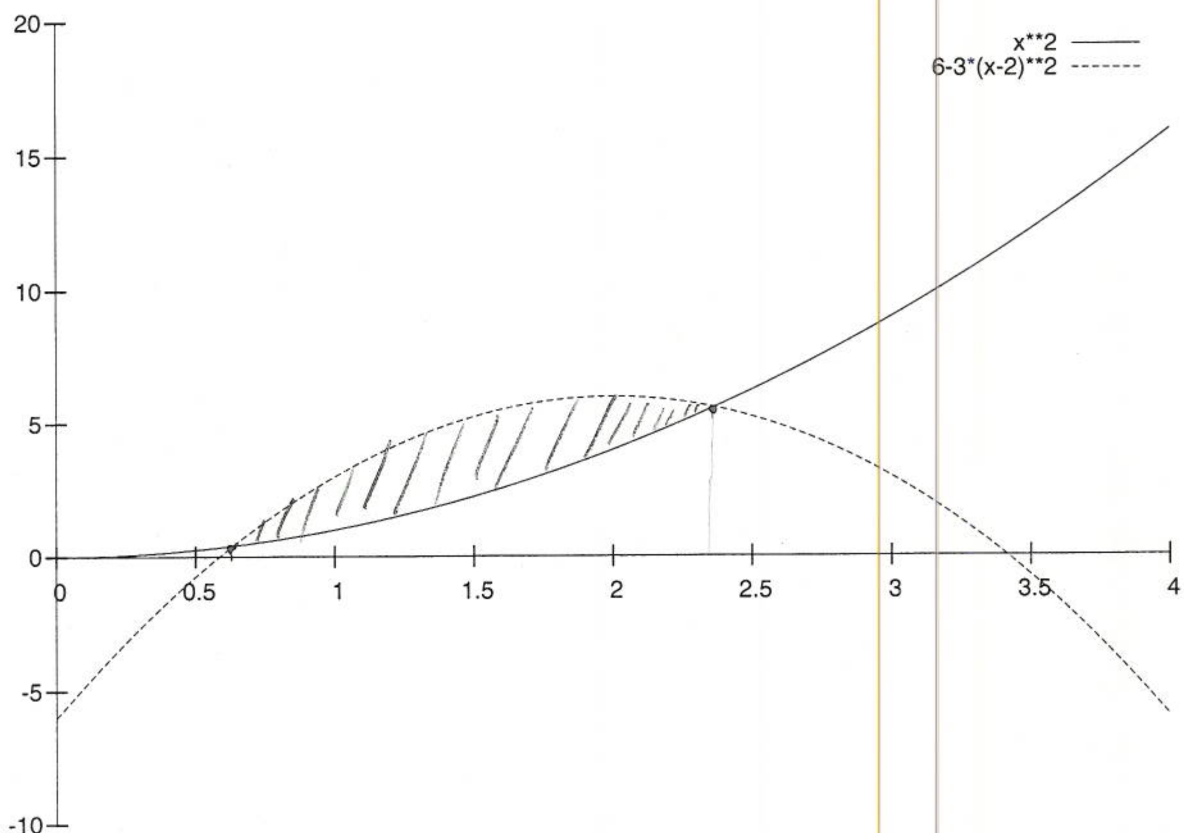
$$= -\ln|\cos(x)| + C$$

(but again, you can only integrate $\tan(x)$ on intervals on which it is defined!)

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

Area between curves

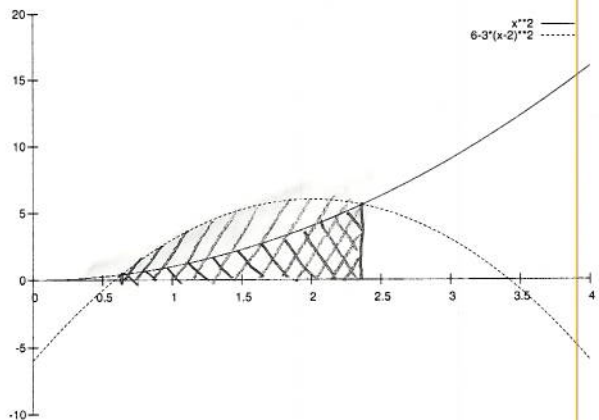
Example: Find the area of the region enclosed by the graphs of x^2 and $6 - 3\,(x-2)^2$.



Solution: We first find the x-values of the intersection points :

$$\begin{aligned} & x^2 = 6 - 3\,(x-2)^2 \\ \Leftrightarrow\ & 4x^2 - 12x + 6 = 0 \\ \Leftrightarrow\ & x = \frac{6 \pm \sqrt{12}}{4} \\ \Leftrightarrow\ & x = 0.634 \text{ or } x = 2.37 \end{aligned}$$

Then the area between the graphs is the difference between the area between top one and the x-axis and the area between the bottom one and the x-axis. So the area is



$$\int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} \left(6-3\,(x-2)^2\right) dx - \int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} x^2 dx = \int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} \left(6-3\,(x-2)^2 - x^2\right) dx$$

$$= \int_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}} \left(-4x^2+12x-6\right) dx$$

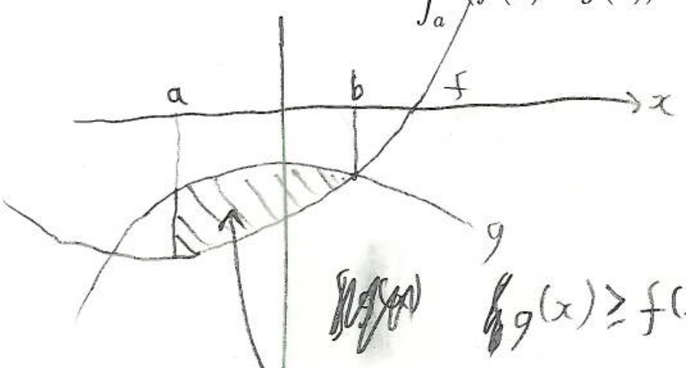
$$= \left[-\frac{4x^3}{3}+6x^2-6x\right]_{\frac{6-\sqrt{12}}{4}}^{\frac{6+\sqrt{12}}{4}}$$

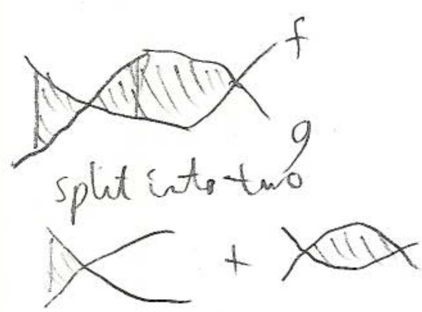
$$= 3.46$$

It wasn't important that the area was above the x-axis, and so we get in general:

Formula: If f and g are continuous functions on $[a,b]$, and if $f(x) \geq g(x)$ on $[a,b]$, then the area of the region enclosed by the graphs of $f(x)$, $g(x)$ and the lines $x=a$ and $x=b$ is

$$\int_a^b (f(x)-g(x))\,dx$$

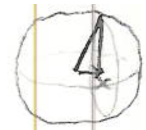




$g(x) \geq f(x)$ on $[a,b]$

$$\int_a^b (g(x)-f(x))\,dx = \int_a^b ((g(x)+c)-(f(x)+c))\,dx$$

Volumes



Example - Volume of a sphere: Consider a sphere of radius r, centred at the origin (0,0,0).

Chop it perpendicular to the x-axis into n slivers of equal width.

The volume of the sphere is the sum of the volumes of the slivers.

For large n, i.e. for thin slivers, each sliver is roughly a cylinder of width $\Delta_n = \frac{2r}{n}$. The radius depends on x: the i^{th} sliver has radius $\sqrt{r^2 - (x_i^*)^2}$ on its right face, where $x_i^* = -r + i\Delta_n$. $(i = 1, \ldots, n)$

So we can estimate the volume of the i^{th} sliver as

$$\Delta_n \pi \left(\sqrt{r^2 - (x_i^*)^2} \right)^2 = \Delta_n \pi \left(r^2 - (x_i^*)^2 \right)$$

So our estimate for the volume with n slivers is

$$V_n = \sum_{i=1}^{n} \Delta_n \pi \left(r^2 - (x_i^*)^2 \right).$$

As $n \to \infty$, our estimates converge to the actual volume. So the volume of the sphere is

$$\begin{aligned}
V &= \lim_{n\to\infty} V_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_n \pi \left(r^2 - (x_i^*)^2 \right) \\
&= \int_{-r}^{r} \pi \left(r^2 - x^2 \right) dx \\
&= \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^{r} \\
&= \pi \left(\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 - \frac{-r^3}{3} \right) \right) \\
&= \frac{4\pi r^3}{3}.
\end{aligned}$$

~~**More examples:** TODO - work in $\int \frac{1}{x} dx$?~~