# Gaussian Elimination (§1.2)

Represent a system of linear equations

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \end{aligned}$$

by the augmented matrix

$$\left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \end{array}\right)$$

(the vertical line is optional)

e.g.

$$\begin{aligned} 3x + 4y - 5z &= 2 \\ x \phantom{+4y} + z &= 5 \\ 8y &= 0 \end{aligned}$$

has augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 4 & -5 & 2 \\ 1 & 0 & 1 & 5 \\ 0 & 8 & 0 & 0 \end{array}\right)$$

An augmented matrix is in reduced row echelon form (rref) if it is of the following form:

leading 1's

$$\left(\begin{array}{ccccccc|c} 0 & 1 & * & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

where the $*$'s stand for arbitrary numbers

## Precise definition:

(i) Each row either consists of zeros, or has first non-zero entry 1 – the leading 1 of the row.

(ii) Each row after the first starts with ~~at least as~~ more zeros than the row before

(iii) A column containing a leading 1 has zeros everywhere else.

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We use <u>elementary row operations</u> to transform an augmented matrix into an rref augmented matrix:

(I) Multiply a row by a non-zero constant

(II) Swap two rows

(III) Add a multiple of one row to a different row

<u>Example</u>: $2x + 3y = 4$
$4x - y = 15$

augmented matrix: $\left(\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & -1 & 15 \end{array}\right)$

$\downarrow$ add $(-2)$-times $1^{st}$ row to $2^{nd}$

$\left(\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & -7 & 7 \end{array}\right)$

$\downarrow$ multiply $1^{st}$ row by $\frac{1}{2}$

$\left(\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & -7 & 7 \end{array}\right)$

$\downarrow$ multiply $2^{nd}$ by $-1/7$

$\left(\begin{array}{cc|c} 1 & 3/2 & 2 \\ 0 & 1 & -1 \end{array}\right)$

$\downarrow$ add $-3/2$ times $2^{nd}$ to $1^{st}$

$\begin{aligned} x &= 7/2 \\ y &= -1 \end{aligned} \longleftrightarrow \left(\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & -1 \end{array}\right)$

<u>The point</u>: If we apply an elementary row operation to the augmented matrix of a linear system, then the

linear system corresponding to the resulting augmented matrix has the same solutions as the original system.

## Describing the solutions of a linear system corresponding to a rref augmented matrix

- If we have a row of form

$$\left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

this corresponds to

$$0 = 1$$

so the system has no solutions.

- Otherwise:

If we pick arbitrary value for the variables corresponding to columns not containing a leading 1

then there is a unique choice of values for the other values giving a solution, and that choice can be read off from the matrix.

$$\begin{array}{cccc} x & y & z & w \\ \end{array} \quad \left(\begin{array}{cccc|c} 0 & 1 & 5 & 0 & 27 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$y + 5z = 27$$
$$w = 1$$
$$0 = 0$$

say $x = s$, $z = t$

then we have to have $y = 27 - 5t$
$w = 1$

$$\boxed{\text{so solutions are } (s, 27-5t, t, 1)}$$

where $s$ and $t$ are arbitrary