## Gaussian Elimination

To transform an augmented matrix into a rref matrix, do the following;

**Stage I;**

**Step I;** Swapping rows (row op II) as necessary, make sure that the first non-zero column doesn't start with 0.
If there are no non-0 columns, proceed to Stage II

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 2 & 0 & 8 & 4 \\ 1 & 0 & 1 & -1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 8 & 4 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right)$$

**Step II;**
Divide the first row through (row op I) such that it has a leading 1

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right)$$

**Step III**
Add multiples of the first row to the lower rows (row op III) such that the leading 1 of the first row has zeros below it

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right)$$

**Step IV:**
Cover up the first row, and repeat from Step I with what remains

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right)$$

**Stage II;** Uncover any covered rows.

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The matrix we have after stage I satisfies (i) and (ii) in the definiction of rref but although the entries <u>below</u> any leading 1 are 0 there may be non-0 entries above (such matrices are called <u>row echelon form</u> matrices)

- Add multiples of the last non-zero row to the rows above to make the entries above the leading 1 of the row 0.
- Repeat with the second-to-last non-zero row, and so on moving up.
- We end up with a 'rref' matrix.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x = -2$

$z = 1$

<u>Example:</u>

$$\begin{aligned} 3x + 5y + 11z - 47w &= 1 \\ x + y + z + w &= 2 \\ 4y - 12z + 4w &= 4 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 3 & 5 & 11 & -47 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 4 & -12 & 4 & 4 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 5/3 & 11/3 & \frac{-47}{3} & 1/3 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 4 & -12 & 4 & 4 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 5/3 & 11/3 & \frac{-47}{3} & 1/3 \\ 0 & -2/3 & -8/3 & 50/3 & 5/3 \\ 0 & 4 & -12 & 4 & 4 \end{array}\right)$$

multiply 2nd row by $-3/2$

$$\left(\begin{array}{cccc|c} 1 & 5/3 & 11/3 & \frac{-47}{3} & 1/3 \\ 0 & 1 & 4 & -25 & -5/2 \\ 0 & 4 & -12 & 4 & 4 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 5/3 & 11/3 & \frac{-47}{3} & 1/3 \\ 0 & 1 & 4 & -25 & -5/2 \\ 0 & 0 & -28 & 104 & 14 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 5/3 & 11/3 & -47/3 & 1/3 \\ 0 & 1 & 4 & -25 & -5/2 \\ 0 & 0 & 1 & \frac{-104}{28} & -1/2 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{\sim}{\ } & \sim \\ 0 & 1 & 0 & \sim & \sim \\ 0 & 0 & 1 & \sim & \sim \end{array}\right) \end{array}$$

$w = t$

$x + 37w = 28$

$x = 28 - 37t$