Example of Stage 2:

After stage 1, we have a ref mat (not necessarily rref)

e.g. $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$ corresponding to

$$\begin{aligned} x + 2y + 3z + 4w &= 5 \\ y + 2z + 3w &= 4 \\ w &= 2 \end{aligned}$$

stage 2 reduces it to rref:

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \underline{\text{rref}}$$

$$\longleftrightarrow \quad \begin{aligned} x \quad - z \quad &= 1 \\ y + 2z \quad &= -2 \\ w &= 2 \end{aligned}$$

solutions are

$(1+t, -2-2t, t, 2)$

Matrices

A *matrix* is a rectangular array of numbers

if it has m rows and n columns, we call it an $m \times n$ matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

2×3 matrix

Augmented matrices are matrices
- ignore vertical line.

$n \times n$ matrices are *square* matrices

$$\begin{pmatrix} 1 & 7 \\ -2 & 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 5 & 2 \\ 0 & 0 & 0 \\ 1 & 8 & -1 \end{pmatrix}$$

Matrices are commonly denoted by capital letters $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

A_{ij} is the "$(i\,j)^{th}$ entry" of the matrix A i.e. that on the i^{th} row and the j^{th} column

e.g. if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

then $A_{11} = 1 \quad A_{12} = 2$

$A_{21} = 3, \quad A_{22} = 4$

If $a_{11}, a_{12}, a_{13}, \ldots, a_{1n}, a_{21}, a_{22}, \ldots, a_{2n}, \ldots, a_{m1}, a_{m2}, \ldots, a_{mn}$ are numbers

($(m)(n)$ of them)

then $[a_{ij}]$ is the matrix with $(i,j)^{th}$ entry a_{ij}

i.e. $([a_{ij}])_{ij} = a_{ij}$

e.g. if $a_{11} = 1$, $a_{12} = 7$

$a_{21} = -5$, $a_{22} = \pi$

then $[a_{ij}] = \begin{pmatrix} 1 & 7 \\ -5 & \pi \end{pmatrix}$

Matrices A B are equal ($A = B$)

~~iff~~ if they are of the same size

and all entries are equal

$A_{ij} = B_{ij}$ for all i,j

A <u>column vector</u> is a ~~matrix~~ $m \times 1$ matrix

e.g. $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

a <u>row vector</u> is a $1 \times n$ matrix

e.g. $(3, 2, 1)$

we denote these by lower case bold or underlined letters e.g. $\underline{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

and then b_i is the i^{th} entry of $\underline{b}$

$b_1 = 3$, $b_2 = 2$, $b_1 = 1$

Matrix addition:

same size

If A and B are $m \times n$ matrices, then $A+B$ is the $m \times n$ matrix with entries

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 7 & 8 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 10 & 12 \\ 7 & 5 \end{pmatrix}$

Scalar multiplication:

If A is an $m \times n$ matrix and c is a number then cA is the $m \times n$ matrix with entries

$$(cA)_{ij} = c A_{ij}$$

e.g. $7\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 14 & 21 \\ 7 & 49 \end{pmatrix}$

Matrix multiplication

Special case: If A is an $m \times n$ matrix and $\underline{b}$ is an $n \times 1$ column vector then we define $A\underline{b}$ to be the $m \times 1$ column vector with entries

$$(A\underline{b})_i = A_{i1} b_1 + A_{i2} b_2 + \dots + A_{in} b_n$$

e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1*1 + 2*2 \\ 3*1 + 4*2 \\ 5*1 + 6*2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 17 \end{pmatrix}$

$3\times 2 \quad 2\times 1 \quad 3\times 1$

If we have a system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = c_1$$
$$a_{21}x + a_{22}y + a_{23}z = c_2$$

if we define $A = [a_{ij}]$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

then we can write the system as

$$A\underline{x} = \underline{c}$$

Now if A is an $m\times n$ matrix
B is an $n\times k$ matrix
$\underline{c}$ is a $k\times 1$ column vector

then $A(\underbrace{B\underline{c}}_{n\times 1 \text{ column vector}})$ makes sense and is an $m\times 1$ column vector

we define AB such that

$$(AB)\underline{c} = A(B\underline{c}) \quad \text{for any } \underline{c}$$

<u>Definition</u>. If A is an $m\times n$ matrix
and B is an $n\times k$ matrix
then the <u>product</u> AB is the $m\times k$ matrix with entries

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{in}B_{nj}$$

Remark: this agrees with the def^n for column vectors

Fact: $A(BC) = (AB)C$ whenever ~~this~~ the sizes are such that the products are defined.

Suppose we have the following systems of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= y_1 \\ 2x_1 + 3x_2 \quad &= y_2 \end{aligned}$$

$$\begin{aligned} y_1 + y_2 &= 3 \\ 3y_1 - 2y_2 &= 6 \end{aligned}$$

We want to solve for x_1, x_2, x_3

i.e. find which values of x_1, x_2, x_3 make the second system hold when y_1, y_2 are set by the first system.

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

so we have $A\underline{x} = \underline{y}$ and $B\underline{y} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

i.e. $B(A\underline{x}) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ i.e. $(BA)\underline{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

so if we calculate BA

we'll have a single linear system in $\underline{x}$

1*1 + 1*2

$$(BA) = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 5 & -1 \\ -1 & 0 & -3 \end{pmatrix}$$

3*2 + (-2*3)

<u>Remark</u>: The procedure we gave for Gaussian elimination is sometimes called "Gauss-Jordan" elimination

<u>Definitions</u>:

- The <u>transpose</u> of an $m \times n$ matrix A is the $n \times m$ matrix A^T with entries

$$(A^T)_{ij} = A_{ji} \qquad \text{e.g.} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

- The <u>trace</u> of an <u>$n \times n$</u> matrix A is the number

$$tr(A) = A_{11} + A_{22} + \ldots + A_{nn}$$

$$tr\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right) = 1+5+9 = 15$$

trace is undefined for matrices which aren't square

Matrix Algebra

Theorem: If A, B and C are matrices, then

(a) $A+B = B+A$

(b) $(A+B)+C = A+(B+C)$

(c) $A(B+C) = AB+AC$

(d) $(A+B)C = AC+BC$

(e) $A(BC) = (AB)C$

when the sizes are such that the operations are defined

Remarks:

Following are not true in general;

- $AB = BA$ e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

  $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

- $AB = AC \Rightarrow B = C$ e.g. A, B as above

  $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

  $AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

  so $AB = AC$

  but $B \neq C$

- $BA = CA \Rightarrow B = C$ (same counter example)

Proof: (a), (b): Easy

(c):
$$\begin{aligned}(A(B+C))_{ij} &= A_{i1}(B+C)_{1j} + \dots + A_{in}(B+C)_{nj} \\ &= A_{i1}(B_{1j}+C_{1j}) + \dots + A_{in}(B_{nj}+C_{nj}) \\ &= A_{i1}B_{1j} + A_{i1}C_{1j} + \dots + A_{in}B_{nj} + A_{in}C_{nj} \\ &= (AB)_{ij} + (AC)_{ij} \\ &= (AB+AC)_{ij}\end{aligned}$$

(d) Similar

(e) Omitted $\square$

Definitions:

- For any m and n, the <u>zero matrix</u> of size $m \times n$, written $0_{m\times n}$ or 0 is the $m\times n$ matrix with all entries 0

$$(0)_{ij} = 0 \qquad \text{e.g. } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{2\times 3}$$

- For any n, the <u>identity matrix</u> of size $n \times n$, written I_n or just I has entries

$$(I)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{e.g. } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Lemma</u>: For any A

(a) $A + 0 = A = 0 + A$

(b) $A0 = 0$

(c) $0A = 0$

(d) $AI = A$

(e) $IA = A$

where 0 is a zero matrix of a size such that the operations are defined

e.g.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

e.g. $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$