Example of Stage 2:

After stage 1, we have a ref matrix (not necessarily rref)

e.g. $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$ corresponding to $\begin{aligned} x+2y+3z+4w&=5 \\ y+2z+3w&=4 \\ w&=2 \end{aligned}$

stage 2 reduces it to rref:

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$ rref

$$\longleftrightarrow \quad \begin{aligned} x \quad -z \quad &= 1 \\ y+2z \quad &= -2 \\ w &= 2 \end{aligned}$$

solutions are

$(1+t, -2-2t, t, 2)$

Matrices

A <u>matrix</u> is a rectangular array of numbers

if it has m rows
and n columns,
we call it an $m \times n$ matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

2×3 matrix

Augmented matrices are matrices
– ignore vertical line.

$n \times n$ matrices are <u>square</u> matrices

$$\begin{pmatrix} 1 & 7 \\ -2 & 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 5 & 2 \\ 0 & 0 & 0 \\ 1 & 8 & -1 \end{pmatrix}$$

Matrices are commonly denoted
by capital letters $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

A_{ij} is the "$(i,j)^{th}$ entry" of the matrix A
i.e. that on the i^{th} row and the j^{th} column

e.g. if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

then $A_{11} = 1 \quad A_{12} = 2$
$A_{21} = 3 \quad , A_{22} = 4$

If $a_{11}, a_{12}, a_{13}, \dots, a_{1n}, a_{21}, a_{22}, \dots, a_{2n}, \dots, a_{m1}, a_{m2}, \dots, a_{mn}$ are numbers

($(m)(n)$ of them)

then $[a_{ij}]$ is the matrix with $(i,j)^{th}$ entry a_{ij}

i.e. $([a_{ij}])_{ij} = a_{ij}$

e.g. if $a_{11} = 1$, $a_{12} = 7$

$a_{21} = -5$, $a_{22} = \pi$

then $[a_{ij}] = \begin{pmatrix} 1 & 7 \\ -5 & \pi \end{pmatrix}$

Matrices A B are equal $(A = B)$

~~iff~~ if they are of the same size

and all entries are equal

$A_{ij} = B_{ij}$ for all i, j

A column vector is a ~~matrix~~ $m \times 1$ matrix

e.g. $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

a row vector is a $1 \times n$ matrix

e.g. $(3, 2, 1)$

we denote these by lower case bold or underlined

letters e.g. $\underline{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

and then b_i is the i^{th} entry

of $\underline{b}$

$b_1 = 3$, $b_2 = 2$, $b_1 = 1$

Matrix addition;

If A and B are $m \times n$ matrices, <u>same size</u>

then $A+B$ is the $m \times n$ matrix with entries;

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 7 & 8 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 10 & 12 \\ 7 & 5 \end{pmatrix}$

Scalar multiplication;

If A is an $m \times n$ matrix

and c is a number

then cA is the $m \times n$ matrix with entries

$$(cA)_{ij} = c A_{ij}$$

e.g. $7\begin{pmatrix} 2 & 3 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 14 & 21 \\ 7 & 49 \end{pmatrix}$

Matrix multiplication

<u>Special case</u>; If A is an $m \times n$ matrix

and $\underline{b}$ is an $n \times 1$ column vector

then we define $A\underline{b}$ to be the $m \times 1$

column vector with entries

$$(A\underline{b})_i = A_{i1} b_1 + A_{i2} b_2 + \dots + A_{in} b_n$$

e.g. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1*1 + 2*2 \\ 3*1 + 4*2 \\ 5*1 + 6*2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 17 \end{pmatrix}$

3×2 $\quad 2\times 1$ $\quad 3\times 1$

If we have a system of linear equations

$$a_{11}x + a_{12}y + a_{13}z = c_1$$
$$a_{21}x + a_{22}y + a_{23}z = c_2$$

if we define $A = [a_{ij}]$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

then we can write the system as

$$A\underline{x} = \underline{c}$$

Now if A is an $m \times n$ matrix

B is an $n \times k$ matrix

$\underline{c}$ is a $k \times 1$ column vector

then $A(\underbrace{B\underline{c}}_{n\times 1 \text{ column vector}})$ makes sense and is an $m \times 1$ column vector

we define AB such that

$$(AB)\underline{c} = A(B\underline{c}) \quad \text{for any } \underline{c}$$

<u>Definition</u>. If A is an $m\times n$ matrix and B is an $n \times k$ matrix then the <u>product</u> AB is the $m \times k$ matrix with entries

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{in}B_{nj}$$

Remark: this agrees with the def^n for column vectors

Fact: $A(BC) = (AB)C$ whenever ~~the~~ the sizes are such that the products are defined.