$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \dots + A_{in}B_{nj}$$

<u>Remark</u>: this agrees with the def^n for column vectors

<u>Fact</u>: $A(BC) = (AB)C$ wherever the sizes are such that the products are defined.

---

suppose we have the following systems of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= y_1 \\ 2x_1 + 3x_2 \quad &= y_2 \end{aligned}$$

$$\begin{aligned} y_1 + y_2 &= 3 \\ 3y_1 - 2y_2 &= 6 \end{aligned}$$

We want to solve for x_1, x_2, x_3
i.e find which values of x_1, x_2, x_3
make the second system hold
when y_1, y_2 are set by the first system.

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

so we have $A\underline{x} = \underline{y}$ and $B\underline{y} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

i.e. $B(A\underline{x}) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ i.e. $(BA)\underline{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

so if we calculate BA,

we'll have a single linear system in $\underline{x}$

1*1+ 1*2

$$(BA) = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 5 & -1 \\ -1 & 0 & -3 \end{pmatrix}$$

3*2 + (-2*3)

<u>Remark</u>: The procedure we gave for Gaussian elimination is sometimes called "Gauss-Jordan" elimination

<u>Definitions</u>:

- The <u>transpose</u> of an $m \times n$ matrix A is the $n \times m$ matrix A^T with entries

$$(A^T)_{ij} = A_{ji} \qquad \text{e.g.} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

- The <u>trace</u> of an <u>$n \times n$</u> matrix A is the number

$$tr(A) = A_{11} + A_{22} + \dots + A_{nn}$$

$$tr\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right) = 1+5+9 = 15$$

trace is undefined for matrices which aren't square

Matrix Algebra

Theorem: If A, B and C are matrices, then

$$\left.\begin{array}{ll} (a) & A+B = B+A \\ (b) & (A+B)+C = A+(B+C) \\ (c) & A(B+C) = AB+AC \\ (d) & (A+B)C = AC+BC \\ (e) & A(BC) = (AB)C \end{array}\right\} \text{ when the sizes are such that the operations are defined}$$

Remarks:

Following are not true in general:

- $AB = BA$ e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

  $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

- $AB = AC \Rightarrow B = C$ e.g. A, B as above

  $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

  $AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

  so $AB = AC$

  but $B \neq C$

- $BA = CA \Rightarrow B = C$ (same counter example)

Proof: (a), (b): Easy

(c): $$\begin{aligned}(A(B+C))_{ij} &= A_{i1}(B+C)_{1j} + \dots + A_{in}(B+C)_{nj} \\ &= A_{i1}(B_{1j}+C_{1j}) + \dots + A_{in}(B_{nj}+C_{nj}) \\ &= A_{i1}B_{1j} + A_{i1}C_{1j} + \dots + A_{in}B_{nj} + A_{in}C_{nj} \\ &= (AB)_{ij} + (AC)_{ij} \\ &= (AB+AC)_{ij}.\end{aligned}$$

(d) similar

(e) omitted $\square$

Definitions:

- For any m and n, the <u>zero matrix</u> of size $m \times n$, written $0_{m\times n}$ or 0 is the $m\times n$ matrix with all entries 0

$$(0)_{ij} = 0 \qquad \text{e.g. } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{2\times 3}$$

- For any n, the <u>identity matrix</u> of size $n \times n$, written I_n or just I has entries

$$(I)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} \qquad \text{e.g. } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Lemma</u>: For any A

(a) $A + 0 = A = 0 + A$

(b) $A0 = 0$

(c) $0A = 0$

(d) $AI = A$

(e) $IA = A$

where 0 is a zero matrix of a size such that the operations are defined

e.g.

$$\begin{pmatrix} 0&0&0 \\ 0&0&0 \end{pmatrix}\begin{pmatrix} 1&4 \\ 2&5 \\ 3&6 \end{pmatrix} = \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$$

e.g. $$\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix}\begin{pmatrix} 1&4 \\ 2&5 \\ 3&6 \end{pmatrix} = \begin{pmatrix} 1&4 \\ 2&5 \\ 3&6 \end{pmatrix}$$