$$
\begin{aligned}
(A(BC))_{ij} &= A_{i1}(BC)_{1j} + \quad + A_{in}(BC)_{nj} \\
&= A_{i1}(B_{11}C_{1j} + \dots + B_{1m}C_{mj}) + \dots + A_{in}(B_{n1}C_{1j} + \dots + B_{nm}C_{mj}) \\
&= (A_{i1}B_{11} + \dots + A_{in}B_{n1})C_{1j} + \dots + (A_{i1}B_{1m} + \quad + A_{in}B_{nm})C_{mj} \\
&= (AB)_{i1}C_{1j} + \dots + (AB)_{im}C_{mj} \\
&= ((AB)C)_{ij}
\end{aligned}
$$

so $\boxed{A(BC) = (AB)C}$

Inverses:

Definition: If A is an $n \times n$ matrix

We say B is an inverse of A

if $AB = I = BA$

If such an inverse exists, we say A is invertible

If A is not invertible, A is singular

Fact: If A has an inverse then it is unique

We then write A^{-1} ("A inverse") for this matrix

$$\boxed{AA^{-1} = I = A^{-1}A}$$

Fact: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible iff the "determinant of A", $\det(A) = ad - bc$ is not zero and then the inverse is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

e.g. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible since $\underbrace{1*4-2*3}_{=-2} \neq 0$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{1*4-2*3}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

I

Indeed $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

e.g. $\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$ is <u>not</u> invertible $(\det = 6*1 - 2*3 = 0)$

<u>Lemma</u>: If A and B are invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

<u>Proof</u>: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = \ldots = I$ similarly $\square$

<u>Remark:</u> If we have a linear system

$$A\underline{x} = \underline{c}$$

and if A is invertible then

$$\underline{x} = A^{-1}\underline{c} \quad \text{is the only solution}$$

Indeed:

if $A\underline{x} = \underline{c}$, then by multiplying on the left by A^{-1} on both sides of the equation

$$A^{-1}(A\underline{x}) = A^{-1}\underline{c}$$

but $A^{-1}(A\underline{x}) = (A^{-1}A)\underline{x} = I\underline{x} = \underline{x}$

so $\underline{x} = A^{-1}\underline{c}$

and $\underline{x} = A^{-1}\underline{c}$ is a solution of $A\underline{x} = \underline{c}$, since

if $\underline{x} = A^{-1}\underline{c}$ then multiplying on the left by A on both sides of the equation

gives $A\underline{x} = AA^{-1}\underline{c} = \underline{c}$

Elementary Matrices

Fix a number n; we work in this section with matrices with n rows. So $I = I_n$.

Recall that an elementary row operation is an operation on matrices of one of the following forms:

- Swapping two rows
- Multiplying a row by a number $c \neq 0$
- Adding c times one row to a different row.

<u>Definition</u>: An <u>elementary matrix</u> is the result of applying an elementary row operation to I.

Let's write E_r for the elementary matrix of an elementary row operation r

$$E_r = r(I)$$

e.g. If $n=3$ and r is swapping the first two rows, then $E_r = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

or if r is adding 5 times the second row to the third row,

$$E_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

<u>Theorem</u>: Let r be an elementary row operation,

$$E_r = r(I)$$

(i) For any A, $r(A) = E_r A$

(ii) E_r is invertible, and E_r^{-1} is elementary

<u>Proof</u>:

(i) e.g. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$

(proof omitted)