<u>Theorem</u>: Let r be an elementary row operation,

$$E_r = r(I)$$

(i) For any A, $r(A) = E_r A$

(ii) E_r is invertible, and E_r^{-1} is elementary

<u>Proof</u>:

(i) e.g.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

(proof omitted)

$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \cdots$$

(exercise)

(ii) First note that there exists a elementary row operation r^{-1} such that for any A $r^{-1}(r(A)) = A = r(r^{-1}(A))$

Indeed:

(I) if r is multiplying a row by $c \neq 0$ then r^{-1} is multiplying that row by $1/c$

(II) if r is swapping two rows, $r^{-1} = r$

(III) if r is adding c times row i to row $j \neq i$ then r^{-1} is adding $(-c)$-times row i to row j

Now consider $E_{r^{-1}} = r^{-1}(I)$

Then $E_r E_{r^{-1}} = E_r r^{-1}(I) \overset{\text{by (i)}}{=} r(r^{-1}(I)) = I$

similarly $E_{r^{-1}} E_r = I$

so E_r is invertible and $E_r^{-1} = E_{r^{-1}}$

<u>Corollary</u>: An $n \times n$ matrix A is invertible if and only if there are elementary row operations $r_1, r_2, \dots, r_k$ such that

$r_k(\dots r_2(r_1(A))) = I$ result of applying the row operations in order to A

in this case,

$$A^{-1} = E_{r_k} E_{r_{k-1}} \dots E_{r_2} E_{r_1} = r_k(\dots r_2(r_1(I))\dots)$$

<u>Proof</u>: By Gauss-Jordan, there are $r_1, \dots, r_k$ such that $r_k(\dots r_2(r_1(A))\dots) = R$ is rref

Let $E := E_{r_k} E_{r_{k-1}} \dots E_{r_2} E_{r_1}$.

Then by the previous theorem, $EA = R$

and E is invertible (since each E_{r_i} is invertible and the product of invertible matrices is invertible)

Now A is invertible if and only if R is

(if A is invertible then $R = EA$ is a product of invertible matrices

and if R is invertible then $A = E^{-1}R$ is invertible)

But R is rref, so R is invertible if and only if $R=I$

(indeed; if $R \neq I$ then the last row of R is zero

but then for any C, the last row of RC is zero

e.g. $\begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $RC \neq I$ so R is not invertible!

So A is invertible $\Leftrightarrow$ $EA = I$

and then $EAA^{-1} = IA^{-1}$

i.e. $E = A^{-1}$

$\square$

Using this to calculate inverses of matrices

Given a square matrix A
apply Gauss-Jordan to A and simultaneously apply the same operations to the identity;
A is invertible iff we get rref I
and then A^{-1} is what I is transformed into

e.g. $\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right)$ $\quad$ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^{-1}$

$\downarrow$

$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 7 & 8 & 4 & 0 & 0 & 1 \end{array}\right) \to \cdots \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \end{array}\right)$