

Theorem: Let r be an elementary row operation,

$$E_r = r(I)$$

(i) For any A , $r(A) = E_r A$

(ii) E_r is invertible, and E_r^{-1} is elementary

Proof:

(i) e.g.
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

(proof omitted)
$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 11 & 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \dots$$

(exercise)

(ii) First note that there exists a elementary row operation r^{-1}

such that for any A $r^{-1}(r(A)) = A = r(r^{-1}(A))$

Indeed:

(I) if r is multiplying a row by $c \neq 0$
then r^{-1} is multiplying that row by $1/c$

(II) if r is swapping two rows,
 $r^{-1} = r$

(III) if r is adding c times row i to row $j \neq i$
then r^{-1} is adding $(-c)$ times row i to row j

Now consider $E_{r^{-1}} = r^{-1}(I)$ by (i)

$$\text{Then } E_r E_{r^{-1}} = E_r r^{-1}(I) = r(r^{-1}(I)) = I$$

$$\text{similarly } E_{r^{-1}} E_r = I$$

so E_r is invertible and $E_r^{-1} = E_{r^{-1}}$

Corollary: An $n \times n$ matrix A is invertible

if and only if there are elementary row operations $r_1, r_2, \dots, r_k$ such that

$$r_k(\dots r_2(r_1(A))) = I$$

result of applying the row operations in order to A

in this case,

$$A^{-1} = E_{r_k} E_{r_{k-1}} \dots E_{r_2} E_{r_1} = r_k(\dots r_2(r_1(I)))$$

Proof: By Gauss-Jordan, there are $r_1, \dots, r_k$ such that $r_k(\dots r_2(r_1(A))) = R$ is rref

$$\text{Let } E := E_{r_k} E_{r_{k-1}} \dots E_{r_2} E_{r_1}$$

Then by the previous theorem, $EA = R$

and E is invertible (since each E_{r_i} is invertible and the product of invertible matrices is invertible)

Now A is invertible if and only if R is

(if A is invertible then $R = EA$ is a product of invertible matrices
and if R is invertible then $A = E^{-1}R$ is invertible)

But R is rref, so R is invertible if and only if $R=I$
 (indeed: if $R \neq I$ then the last row of R is zero

but then for any C , the last row of RC is zero

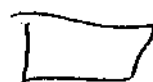
e.g. $\begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

so $RC \neq I$ so R is not invertible!

So A is invertible $\Leftrightarrow EA = I$

and then $EAA^{-1} = IA^{-1}$

i.e. $E = A^{-1}$



Using this to calculate inverses of matrices

Given a square matrix A

apply Gauss-Jordan to A and simultaneously apply the same operations to the identity;

A is invertible iff we get rref ~~A~~ I
 and then A^{-1} is what I is transformed into

e.g. $\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array} \right)$



$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array} \right) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dots & \dots & \dots \\ 0 & 1 & 0 & \dots & \dots & \dots \\ 0 & 0 & 1 & \dots & \dots & \dots \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array} \right)^{-1}$$

