$A\underline{x} = \underline{b}$

The nicest behaviour is when for any $\underline{b}$ there is a unique solution.

We have seen that if A is square (i.e. n equations in n unknowns) if A is invertible, then we have this nice behaviour $\underline{x} = A^{-1}\underline{b}$

Two things can go wrong: we can have $\underline{b}$ for which $A\underline{x} = \underline{b}$ has no solution or we can have $\underline{b}$ such that we have more than one solution.

<u>Example with A singular:</u>

$$
\begin{aligned}
x + 2y + z &= b_1 \\
2x + y + 2z &= b_2 \\
x + y + z &= b_3
\end{aligned}
$$

$$
\overbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}}^{A}
\overbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}^{\underline{x}}
=
\overbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}^{\underline{b}}
$$

$A\underline{x} = \underline{b}$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\;\;2\;\;1}}^{A}} & & & \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & 0 & -2 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/3 & -1/3 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & -1/3 & -1/3 & 1 \end{array}\right)$$

$$\downarrow$$

$$\Bigg(\underbrace{\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}}_{R}\Bigg|\underbrace{\begin{array}{ccc} -2/3 & 2/3 & 0 \\ 2/3 & -1/3 & 0 \\ -1/3 & -1/3 & 1 \end{array}}_{E}\Bigg) \qquad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$EA = R$ E is invertible

$A = E^{-1}R$ so A is not invertible because $R \neq I$

consider $R\underline{x} = \underline{b}$

$R\underline{x}$ has 0 at the bottom

so e.g. $R\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution

so $EA\underline{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution

so $A\underline{x} = E^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution

$\underline{b} := E^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$A\underline{x} = \underline{b}$ has no solution

$R\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$R\begin{pmatrix} 5 \\ 0 \\ -5 \end{pmatrix} = \underline{0}$

so $R\underline{x} = \underline{0}$ has infinitely many solutions

$R = EA$ so ~~WA~~ if $R\underline{x} = \underline{0}$

then $EA\underline{x} = \underline{0}$

so $A\underline{x} = E^{-1}\underline{0} = \underline{0}$

so $A\underline{x} = \underline{0}$ has infinitely many solutions

Theorem: A square

The following are equivalent:

(a) A invertible

(b) A has rref I

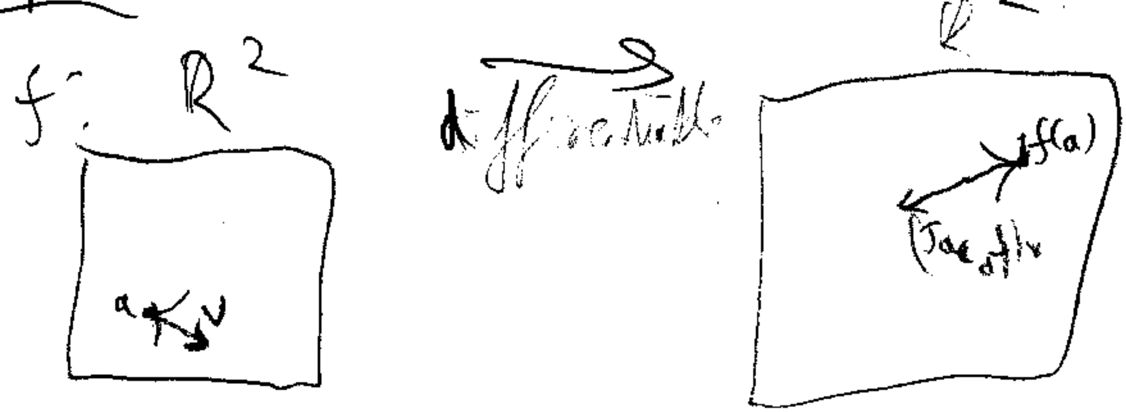
(c) A is a product of elementary matrices

(d) $A\underline{x} = \underline{b}$ has a solution for any $\underline{b}$

(e) $A\underline{x} = \underline{0}$ has only $\underline{x} = \underline{0}$ as a solution

(f) $A\underline{x} = \underline{b}$ has a unique solution for any $\underline{b}$

Example Not examinable ~~Not part of the course~~



If I start at a
I want to ~~m~~ find a direction to move in
such that on the right I move directly upwards (instantaneously)

The Jacobian of f at a $\qquad f(x,y)=(f_1(x,y), f_2(x,y))$
is the matrix

$$\mathrm{Jac}_a f := \begin{pmatrix} df_1/dx & df_1/dy \\ df_2/dx & df_2/dy \end{pmatrix}$$

if $v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector representing velocity of movement from a
then $(\mathrm{Jac}_a f)v$ is the velocity of corresponding movement on the right

So if we can't find v such that

$$(\mathrm{Jac}_a f)\underline{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then there is a $\underline{v} \neq \underline{0}$

$$\text{s.t. } (\mathrm{Jac}_a f)\underline{v} = \underline{0}$$

i.e. a direction to move on the left which (instantaneously) leaves us stationary (on the right)