Diagonal, Triangular and Symmetric §1.7

Definition;

A diagonal matrix is a square matrix D such that $D_{ij} = 0$ if $i \neq j$

$$\underset{\text{Diag}(1,2,3)}{\underset{\|}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}}} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Diagonal

not diagonal

· $\text{Diag}(a_1, \ldots, a_n)$ is the diagonal matrix

$$\begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix} \quad \text{i.e. } \big(\text{Diag}(a_1, \ldots, a_n)\big)_{ii} = a_i$$

· An upper triangular matrix is a square matrix T such that $T_{ij} = 0$ if $i > j$

e.g. $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

· A lower triangular matrix is a square matrix T such that $T_{ij} = 0$ if $i < j$

Remarks;

· $\text{Diag}(a_1, \ldots, a_n)\,\text{Diag}(b_1, \ldots, b_n) = \text{Diag}(a_1 b_1, \ldots, a_n b_n)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

- Stage I of Gauss-Jordan elimination gives an upper triangular matrix

- The product of two upper triangular matrices is upper triangular

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \left(\quad \bullet \right.$$

- " " " lower "

" lower "

<u>Definition</u>: A <u>symmetric</u> matrix is a square matrix A such that $A^T = A$

<u>Recall</u>: $(A^T)_{ij} = A_{ji}$

so symmetric $\Leftrightarrow A_{ij} = A_{ji}$

$$\begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 8 \\ 5 & 8 & 3 \end{pmatrix}$$

<u>Lemma</u>: (i) For any matrix A, $(A^T)^T = A$

(ii) For any matrices A B such that AB is defined

$$(AB)^T = B^T A^T$$

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}} \underset{B}{\begin{pmatrix} 2 & 8 & 1 \\ 7 & 6 & 0 \\ 2 & 3 & 4 \end{pmatrix}} = \begin{pmatrix} \bullet & & \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} 2 & 7 & 2 \\ 8 & 6 & 3 \\ 1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \begin{pmatrix} \cdot & & \\ & & \\ & & \end{pmatrix}$$

$$\quad B^T \qquad\quad A^T$$

(iii) If A is invertible
then so is A^T
and $(A^T)^{-1} = (A^{-1})^T$

(indeed: $(A^{-1})^T A^T = (A A^{-1})^T = I^T = I$
sim. $A^T (A^{-1})^T = I$)

$$\begin{pmatrix} \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$$

<u>Remarks:</u>

- If A and B are symmetric
  then so is $A+B$

- If $AB = BA$ and A, B are symmetric
  then AB is symmetric:
  $$(AB)^T = (BA)^T = A^T B^T = AB$$

- If AB is symmetric and A and B are symmetric
  then $AB = BA$:
  $$BA = B^T A^T = (AB)^T = AB$$

- If A is symmetric and invertible
  then A^{-1} is symmetric
  since $(A^{-1})^T = (A^T)^{-1} = A^{-1}$

Resumé,

Given a linear system $A\underline{x} = \underline{b}$.

Gauss-Jordan yields a sequence of elementary row operations reducing A to an rref matrix R.
Applying these operations to the identity matrix, we obtain an invertible E (product of the corresponding elementary matrices) such that $EA = R$

Now since E is invertible,

$$A\underline{x} = \underline{b} \iff EA\underline{x} = E\underline{b} \iff R\underline{x} = E\underline{b}$$

(using augmented matrices simultaneously computes R and $E\underline{b}$

$$\begin{array}{c} \begin{array}{cc} A & \underline{b} \end{array} \\ \left(\begin{array}{c|c} \cdots & \vdots \end{array} \right) \end{array} \longrightarrow \begin{array}{c} \begin{array}{cc} R & E\underline{b} \end{array} \\ \left(\begin{array}{c|c} \cdots & \vdots \end{array} \right) \end{array}$$

~~If $R \neq I$~~

A is invertible iff $R = I$

so $R\underline{x} = E\underline{b}$

reads $\underline{x} = E\underline{b}$

and $EA = I$ so $E = A^{-1}$

so it just says $x = A^{-1}\underline{b}$ (unique solution for any $\underline{b}$)

· If R has a row of zeroes

then $R\underline{x} = E\underline{b}$ ~~sobmt~~ has no solution for some $\underline{b}$

hence $A\underline{x} = \underline{b}$

· If R has a column not containing a leading 1 then (by setting ~~the~~ the corresponding variable to an arbitrary value)

$R\underline{x} = \underline{0}$ has multiple (infinitely many) solutions

so $Ax = E\underline{0} = \underline{0}$ does too.

If A is square and not invertible, then R has a row of zeroes and a column without a leading 1.