Determinants

Call a selection of cells of a square matrix "acceptable" if each row and column has exactly one selected cell

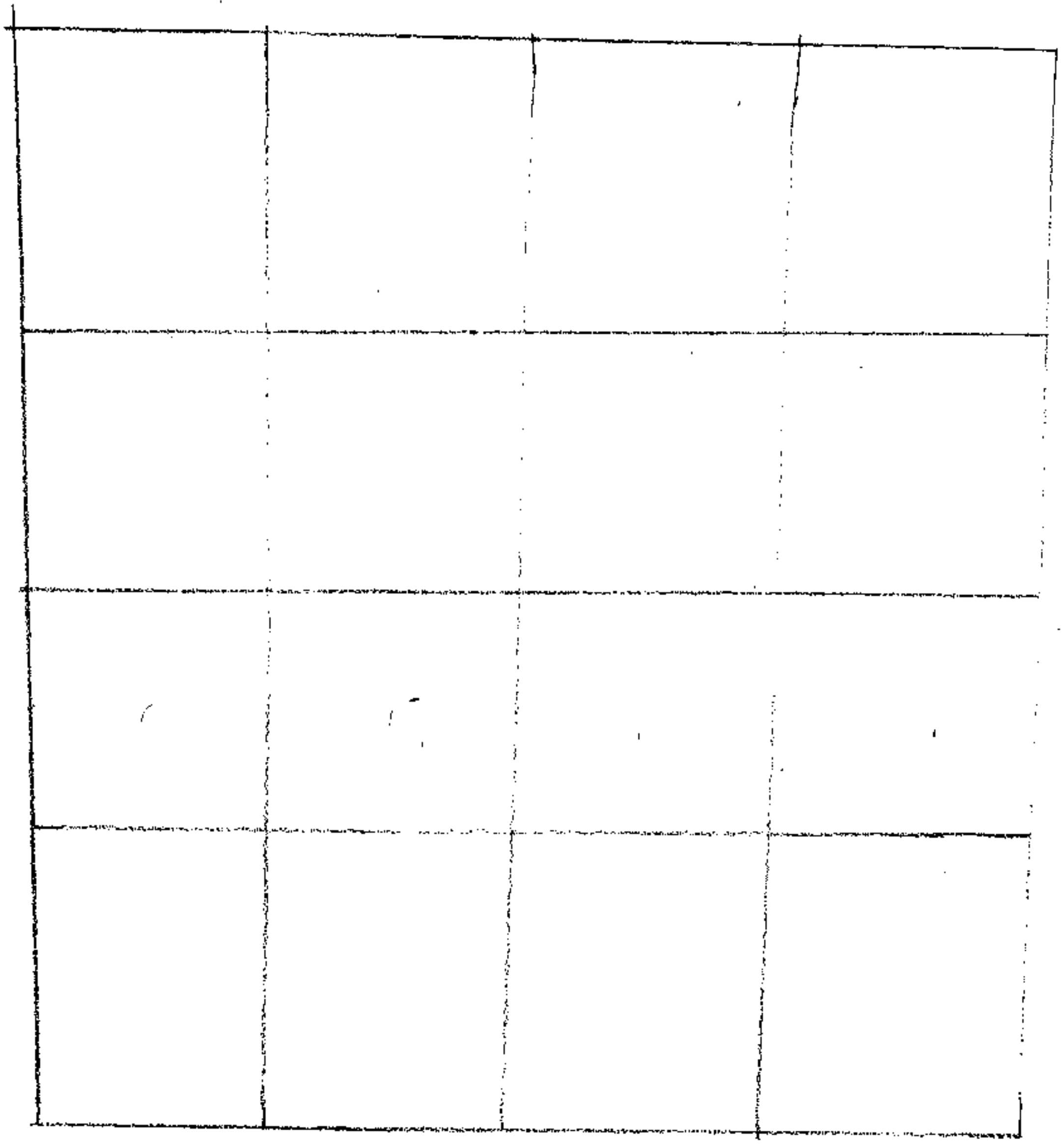
An acceptable selection has a "sign", positive or negative.
The principal diagonal (top left to bottom right) is positive.
If we swap two rows or columns of an acceptable selection, the resulting selection is acceptable and has the opposite sign.

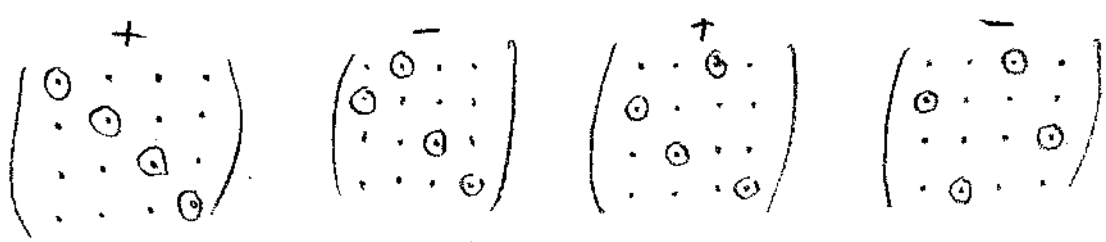
(This makes sense! (Not entirely obvious))

- The "signed product" of an acceptable selection is the product of the selected entries if the sign of the selection is +ve or minus the product of the selected entries if ~~the~~ sign is -ve

- The <u>determinant</u> of a square matrix is the sum of the signed products of the acceptable selections.

e.g. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$





$$\begin{array}{cccccc} + & - & - & + & + & - \\ \begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} & \begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} & \begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} & \begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} & \begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} & \begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} \\ 6 & -1 & -8 & 6 & 6 & -27 \end{array}$$

signed products

so $\det\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix} = 6+(-1)+(-8)+6+6+(-27)$

$= -12$

Remarks: • $\det(I) = 1$

and $\det(\mathrm{Diag}(a_1, a_2, \dots, a_n)) = a_1 a_2 \dots a_n$

(since only the principle diagonal has non-zero signed product)

$$\det\left(\begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right) = -6$$

• if A has a zero row or column, then $\det(A) = 0$ (since every signed product involves a zero, so is zero)

• $\det(A^T) = \det(A)$

(~~taking the transpose~~ of a selection doesn't change the sign, so we evaluate the same sum in calculating the two determinants)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

• If we swap two rows or columns of A to get B, then $\det(B) = -\det(A)$

(since the signs of the selections flip)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

• In particular, if two rows (or columns) of A are equal, $\det(A) = -\det(A)$

so $\det(A) = 0$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$