

- If we consider varying one row (or column) while keeping the others fixed, the determinant acts linearly i.e.

$$\det \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = c \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \det \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

(similarly for any size of square matrix
and any row or column)

$$\text{e.g. } (a_{11} + b_{11})a_{22}a_{33} = a_{11}a_{22}a_{33} + b_{11}a_{22}a_{33}$$

- In particular $\det(cA) = c^n \det(A)$ (A $n \times n$)
(we're multiplying n rows by c)
- No nice formula for $\det(A+B)$ in terms of $\det(A)$ and $\det(B)$

So we have the following behaviour of $\det$ under elementary row operations:

- If B is the result of swapping two rows of A then $\det(B) = -\det(A)$
- If B is the result of multiplying a row of A by $c \neq 0$ then $\det(B) = c \det(A)$

• If B is the result of adding a multiple of one row of A to a different row,

then $\det(B) = \det(A)$

e.g. $\det \begin{pmatrix} a_{11} + ca_{21} & a_{12} + ca_{22} & a_{13} + ca_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$+ \det \begin{pmatrix} ca_{21} & ca_{22} & ca_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= \det \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ (equal rows)

$= 0$

Calculating determinants by row reduction:

Reduce A to rref R using e.r.o's

Each e.r.o. multiplies the determinant by a non-zero number,

so $\det(R) = e \det(A)$ where e is the product of these numbers. So $e \neq 0$.

If $R = I$, then $\det(R) = 1$ so $\det(A) = 1/e$
 else R has a row of zeros, so $\det(R) = 0$
 so $\det(A) = 0$

So since $R=I$ if and only if A is invertible

Theorem: For any square matrix A ,

$$\boxed{\det(A) = 0 \text{ if and only if } A \text{ is singular.}}$$

Example of calculating $\det(A)$ with row reduction:

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \xrightarrow{\cdot (1/4)} \begin{pmatrix} 1 & 3/4 \\ 2 & 1 \end{pmatrix} \xrightarrow{\cdot 1} \begin{pmatrix} 1 & 3/4 \\ 0 & -1/2 \end{pmatrix}$$

$$\text{so } \det(A) = \det \begin{pmatrix} 1 & 3/4 \\ 0 & -1/2 \end{pmatrix} = -2$$

$$= 1 \cdot (-1/2) = -1/2 = -2$$

$$\downarrow \cdot (-2)$$
$$\begin{pmatrix} 1 & 3/4 \\ 0 & 1 \end{pmatrix}$$

$$\cdot 1 \downarrow$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\cdot (-1)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{so } \det(A) = -1$$

Also:

$$\text{Theorem: } \boxed{\det(AB) = \det(A) \det(B)}$$

A, B $n \times n$ matrices

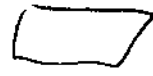
Proof: If A is singular then $Ax=b$ has no solution

for some b , so $ABx=b$ has no solution, so AB is singular

$$\text{so in this case, } \det(AB) = 0 = \det(A) \det(B)$$

If A is not singular, A is a product of elementary

matrices. So A is the result of applying the corresponding row ops to I and AB is the result of applying them to B
so $\det(AB) = \det(A) \det(B)$



(e.g. $\det(ABC) = \det((AB)C) = \det(AB) \det(C)$
 $= \det A \det B \det C$.)