- If we consider varying one row (or column) while keeping the others fixed, the determinant acts <u>linearly</u> i.e.

- $$\det\begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = c \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

- $$\det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$
$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \det\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

(similarly for any size of square matrix and any row or column)

e.g. $(a_{11}+b_{11})a_{22}a_{33} = a_{11}a_{22}a_{33} + b_{11}a_{22}a_{33}$

- In particular $\det(cA) = c^n \det(A)$ (A $n\times n$)
(we're multiplying n rows by c)
- No nice formula for $\det(A+B)$ in terms of $\det(A)$ and $\det(B)$

- So we have the following behaviour of det under elementary row operations:
  - If B is the result of swapping two rows of A then $\det(B) = -\det(A)$
  - If B is the result of multiplying a row of A by $c \neq 0$ then $\det(B) = c\det(A)$

· If B is the result of adding a multiple of one row of A to a different row,

then $\det(B) = \det(A)$

e.g. $\det\begin{pmatrix} a_{11}+ca_{21} & a_{12}+ca_{22} & a_{13}+ca_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$= \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}_{A}$$

$$+ \underbrace{\det\begin{pmatrix} ca_{21} & ca_{22} & ca_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}$$

$$\| $$

$$c\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ equal rows}$$

$$\|$$

$$0.$$

Calculating determinants by row reduction:

Reduce A to rref R using e.r.o's

Each e.r.o. multiplies the determinant by a non-zero number,

so $\det(R) = e\det(A)$ where e is the product of these numbers. So $e \neq 0$.

If $R = I$, then $\det(R) = 1$ so $\det(A) = 1/e$

else R has a row of zeros, so $\det(R) = 0$

so $\det(A) = 0$

So since $R = I$ if and only if A is invertible

<u>Theorem</u>: For any square matrix A,

$$\boxed{\det(A) = 0 \text{ if and only if } A \text{ is singular.}}$$

<u>Example of calculating $\det(A)$ with row reduction</u>:

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \xrightarrow[\cdot(1/4)]{} \begin{pmatrix} 1 & 3/4 \\ 2 & 1 \end{pmatrix} \xrightarrow[\cdot 1]{} \begin{pmatrix} 1 & 3/4 \\ 0 & -1/2 \end{pmatrix} \xrightarrow[]{\cdot(-2)} \begin{pmatrix} 1 & 3/4 \\ 0 & 1 \end{pmatrix} \xrightarrow[\cdot 1]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\det(A) = \cancel{1(-\tfrac{1}{2}) = -2}$

$$= \frac{1}{\left(\frac{1}{4}\cdot -2\right)} = \frac{1}{-\frac{1}{2}} = -2$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow[\cdot(-1)]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\det(A) = -1$

Also:

<u>Theorem</u>: $\boxed{\det(AB) = \det(A)\det(B)}$ A, B $n \times n$ matrices

<u>Proof</u>: If A is singular then $Ax = b$ has no solution for some b, so $ABx = b$ has no solution, so AB is singular so in this case, $\det(AB) = 0 = \det(A)\det(B)$

If A is not singular, A is a product of elementary

matrices. So A is the result of applying the corresponding row ero's to I
and AB is the result of applying them to B

So $\det(AB) = \det(A)\det(B)$

$\square$

(e.g. $\det(ABC) = \det((AB)C) = \det(AB)\det(C)$
$= \det A \det B \det C$.