matrices. So A is the result of applying the corresponding row ero's to I
and AB is the result of applying them to B

So $\det(AB) = \det(A)\det(B)$ $\square$

(e.g. $\det(ABC) = \det((AB)C) = \det(AB)\det(C)$
$= \det A \det B \det C$.

· <u>Remark</u>: If A is invertible,
then $\det(A^{-1}) = \det(A)^{-1} = \frac{1}{\det(A)}$
since $\det(A^{-1})\det(A) = \det(A^{-1}A) = \det(I) = 1$

Cofactor Expansions

Let A be an $n \times n$ matrix

Cofactor expansion on the first row:

Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by crossing out the ith row and jth column (the ijth minor of A)

$$\det(A) = a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}\det M_{13} - \dots + (-1)^{1+n} a_{1n}\det M_{1n}$$

Example:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1\det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 1\cdot(5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7)$$

$$= 0$$

We can do cofactor expansion on any row, but the signs are flipped if we use an even row

so
$$\det(A) = a_{i1}(-1)^{i+1}\det(M_{i1}) + a_{i2}(-1)^{i+2}\det M_{i2} + \dots + a_{in}(-1)^{i+n}\det M_{in}$$

if we let $C_{ij} = (-1)^{i+j}\det(M_{ij})$

we have $\det(A) = a_{i1}C_{i1} + \dots + a_{in}C_{in}$ (for any i)

similarly for columns

$$\det(A) = a_{1i}C_{1i} + \dots + a_{ni}C_{ni} \quad \text{(any } i\text{)}$$

and note if $i \neq j$

then $a_{i1}C_{j1} + \dots + a_{in}C_{jn} = 0$ because it's the determinant of the matrix obtained by replacing the jth row with the ith row so set $\text{cof}(A)$ to be the $n\times n$ matrix with entries

$$(\text{cof}(A))_{ij} = C_{ji}$$

then $\text{cof}(A)\, A = \det(A) I = A\, \text{cof}(A)$

so

Theorem; If A is invertible, then

$$\boxed{A^{-1} = \frac{\text{cof}(A)}{\det(A)}}$$

e.g. 2×2 case : $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\text{cof}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

so $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Don't memorise this!

e.g. 3×3 case : $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\det(A) = a(ei-fh) - b(di-fg) + c(dh-eg)$$

$$\text{cof}(A) = \begin{pmatrix} (ei-fh) & -(bi-ch) & (bf-ce) \\ -(di-fg) & (ai-cg) & -(af-dc) \\ (dh-eg) & -(ah-bg) & (ae-bd) \end{pmatrix}$$

$$A^{-1} = \frac{\text{cof}(A)}{\det(A)} \qquad \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \ (-1)^{i+j}$$

So since $R = I$ if and only if A is invertible

<u>Theorem</u>: For any square matrix A,

$$\boxed{\det(A) = 0 \quad \text{if and only if} \quad A \text{ is singular.}}$$

<u>Example of calculating $\det(A)$ with row reduction</u>:

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \xrightarrow[\cdot(1/4)]{} \begin{pmatrix} 1 & 3/4 \\ 2 & 1 \end{pmatrix} \xrightarrow[\cdot 1]{} \begin{pmatrix} 1 & 3/4 \\ 0 & -1/2 \end{pmatrix} \xrightarrow{\cdot(-2)} \begin{pmatrix} 1 & 3/4 \\ 0 & 1 \end{pmatrix} \xrightarrow{\cdot 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\det(A) = \cancel{4(-1/2) = -2}$

$$= 1/(1/4 \cdot -2) = 1/-1/2 = -2$$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow[\cdot(-1)]{} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\det(A) = -1$

Also:

<u>Theorem</u>: $\boxed{\det(AB) = \det(A)\det(B)}$ $\quad A, B$ $n \times n$ matrices

<u>Proof</u>: If A is singular then $Ax = b$ has no solution for some b, so $ABx = b$ has no solution, so AB is singular so in this case, $\det(AB) = 0 = \det(A)\det(B)$

If A is not singular, A is a product of elementary