Cofactor Expansions

Let A be an $n \times n$ matrix

Cofactor expansion on the first row:

Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by crossing out the ith row and jth column (the ijth minor of A)

$$\det(A) = a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}\det M_{13} - \cdots + (-1)^{1+n} a_{1n}\det M_{1n}$$

Example:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1\det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2\det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 1\cdot(5\cdot 9 - 6\cdot 8) - 2(4\cdot 9 - 6\cdot 7) + 3(4\cdot 8 - 5\cdot 7)$$

$$= 0$$

We can do cofactor expansion on any row, but the signs are flipped if we use an even row

so $$\det(A) = a_{i1}(-1)^{i+1}\det(M_{i1}) + a_{i2}(-1)^{i+2}\det M_{i2} + \cdots + a_{in}(-1)^{i+n}\det M_{in}$$

if we let $C_{ij} = (-1)^{i+j}\det(M_{ij})$

we have $\det(A) = a_{i1}C_{i1} + \cdots + a_{in}C_{in}$ (for any i)

similarly for columns

$$\det(A) = a_{1i}C_{1i} + \cdots + a_{ni}C_{ni} \quad (\text{any } i)$$

and note if $i \neq j$

then $a_{i1}C_{j1} + \cdots + a_{in}C_{jn} = 0$ because it's the determinant of the matrix obtained by replacing the jth row with the ith row

so set adj(A) to be the $n \times n$ matrix with entries
$(\mathrm{adj}(A))_{ij} = C_{ji}$ ← the <u>adjoint</u> matrix

then $\mathrm{adj}(A)\, A = \det(A) I = A\, \mathrm{adj}(A)$

so
<u>Theorem</u>; If A is invertible, then

$$\boxed{A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}}$$

e.g. 2×2 case : $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\mathrm{cof}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

so $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

<u>Don't memorise this!</u>

e.g. 3×3 case : $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\det(A) = a(ei-fh) - b(di-fg) + c(dh-eg)$$

$$\mathrm{cof}(A) = \begin{pmatrix} (ei-fh) & -(bi-ch) & (bf-ce) \\ -(di-fg) & (ai-cg) & -(af-dc) \\ (dh-eg) & -(ah-bg) & (ae-bd \end{pmatrix}$$

$$A^{-1} = \frac{\mathrm{adj}(A)}{\det(A)}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \quad (-1)^{i+j}$$