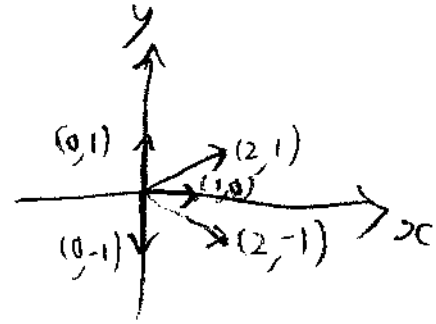
Diagonalisation, eigenvectors, eigenvalues

consider the x-y plane, and consider reflecting in the x-axis, as a map from the plane to itself

so (x,y) is mapped to $(x,-y)$



If we represent a point (x,y) by the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$,

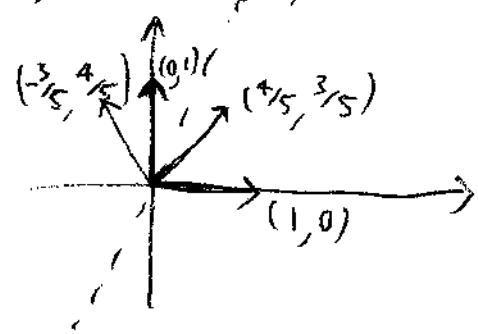
we can represent ~~this~~ map by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

Now consider reflection in the line $y = 2x$, $(1,2)$

$(1,0)$ is mapped to $(-3/5, 4/5)$

$(0,1)$ is mapped to $(4/5, 3/5)$



so the matrix representation is

$$\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$$

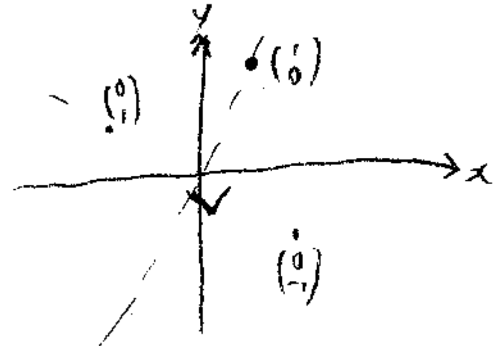
<u>but</u> if we pick new co-ordinates, such that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents the point with x-y co-ordinates $(1,2)$

and such that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents the point with x-y co-ordinates $(-2, 1)$

(which is on the line $y = -x/2$ perpendicular to $y = 2x$)

so then generally $\begin{pmatrix} u \\ v \end{pmatrix} = u\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents the point with x-y co-ordinates $(u-2v, 2u+v)$

then the matrix of our reflection in these new co-ordinates is



$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}u\\-v\end{pmatrix}$$

<u>Generally</u>, a map represented by a diagonal matrix is simple to understand - it just "rescales" in the directions of our co-ordinates

$$\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\lambda_1 x\\\lambda_2 y\\\lambda_3 z\end{pmatrix}$$

A matrix which isnt diagonal might be "diagonalisable" meaning that it represents a map which does have a diagonal represention ~~with~~ for <u>some</u> choice of co-ordinates

<u>Eigenvectors and eigenvalues:</u>

A column vector $\underline{x}$ is an <u>eigenvector</u> of a matrix A if $\underline{x} \neq \underline{0}$ and there is a number λ such that

$$A\underline{x} = \lambda \underline{x}$$

λ is then called the eigenvalue of $\underline{x}$

and λ is an eigenvalue of A

and $\underline{x}$ is a λ-eigenvector

<u>Examples</u>;

$$\overset{A}{\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}=2\begin{pmatrix}0\\1\\0\end{pmatrix}$$

So $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a 2-eigenvector for A

and so is $\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$ $\quad A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix}$

and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a 3-eigenvector

but $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is not an eigenvector

the eigenvalues of A are $1, 2$ and 3

But note that $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

indeed any $\underline{x} \neq \underline{0}$ is

with eigenvalue 1

- $A = \begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$ represents reflection in $y = 2x$

so e.g. $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is a 1-eigenvector since $A\underline{x} = \underline{x} = 1\underline{x}$

as is any non-zero $\underline{x}$ on the line $y = 2x$

and e.g. $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is a -1-eigenvector

as is any non-zero $\underline{x}$ on the perpendicular $y = \frac{-x}{2}$

there are no other eigenvectors